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Noncompact Gaugings, Chiral Reduction and Dual Sigma Models in Supergravity

E. Bergshoeff [†], D.C. Jong ^{*} and E. Sezgin ^{*}

[†] *Institute for Theoretical Physics, Nijenborgh 4, 9747 AG Groningen, The Netherlands*

^{*} *George P. and Cynthia W. Mitchell Institute for Fundamental Physics, Texas A&M
University, College Station, TX 77843-4242, U.S.A*

ABSTRACT

We show that the half-maximal $SU(2)$ gauged supergravity with topological mass term admits coupling of an arbitrary number of n vector multiplets. The chiral circle reduction of the *ungauged* theory in the dual 2-form formulation gives $N = (1, 0)$ supergravity in $6D$ coupled to $3p$ scalars that parametrize the coset $SO(p, 3)/SO(p) \times SO(3)$, a dilaton and $(p+3)$ axions with $p \leq n$. Demanding that R -symmetry gauging survives in $6D$ is shown to put severe restrictions on the $7D$ model, in particular requiring noncompact gaugings. We find that the $SO(2, 2)$ and $SO(3, 1)$ gauged $7D$ supergravities give a $U(1)_R$, and the $SO(2, 1)$ gauged $7D$ supergravity gives an $Sp(1)_R$ gauged chiral $6D$ supergravities coupled to certain matter multiplets. In the $6D$ models obtained, with or without gauging, we show that the scalar fields of the matter sector parametrize the coset $SO(p+1, 4)/SO(p+1) \times SO(4)$, with the $(p+3)$ axions corresponding to its abelian isometries. In the ungauged $6D$ models, upon dualizing the axions to 4-form potentials, we obtain coupling of p linear multiplets and one special linear multiplet to chiral $6D$ supergravity.

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1 Introduction

An impressively large number of string/M theory vacua admit a low energy supergravity description in diverse dimensions in which, however, the R-symmetry group is either trivial or a global (ungauged) symmetry. There exists a far smaller class of vacua which admit gauged supergravities, which, by definition are those in which the R-symmetry is nontrivial and gauged by means of a vector field inside or outside the supergravity multiplet, or a combination thereof¹. These have played an important role in phenomena such as the AdS/CFT as well as the domain-wall/quantum field theory correspondence. The standard examples involve the maximally supersymmetric $SO(N)$ gauged supergravities but it has been shown that the noncompact gauged supergravities, whose gauge group is a contraction and/or analytic continuation of $SO(N)$, with less supersymmetries play a role as well.

While a full classification of all possible gauged supergravities is not available, it is natural to investigate whether all of the ones that are known can be obtained from string/M theory. This is not a simple problem, of course, when addressed in such a generality. However, there is a class of gauged supergravities that are especially interesting to explore, namely matter coupled gauged minimal supergravities in six dimensions [1]. One immediate reason why these are interesting is that the requirement of anomaly freedom turns out to be highly restrictive such that there are very few models that satisfy these criteria [2, 3, 4, 5, 6], and moreover, so far it is not known if any of these models can be embedded in string/M-theory. We expect that if and when such embeddings are discovered, they are likely to reveal novel phenomena in string/M-theory.

Certain progress has already been made by Cvetič, Gibbons and Pope [7] who showed that the $U(1)$ gauged minimal 6D supergravity, while anomalous, it can nonetheless be obtained from M-theory on $H_{2,2} \times S^1$ followed by a truncation. Here, $H_{2,2}$ is a noncompact hyperboloidal 3-manifold that can be embedded in a $(2, 2)$ signature plane. The new phenomenon is that the intermediary 7D theory obtained prior to the circle reduction and chiral truncation must have a noncompact gauge group, in this case $SO(2, 2)$. Naive reduction attempts had not worked prior to this observation.

The purpose of this paper is to start from a general gauged supergravity theory in 7D [13], and putting aside its string/M-theory origin, as well as the issue of anomaly-freedom for now, to look for the most general circle reduction followed by chiral truncation with the criteria that a *gauged* 6D supergravity results. The generic *ungauged* half-maximal supergravity

¹When the gauge group is larger than the R-symmetry group, they will still be referred to as gauged supergravities as long as it contains the R-symmetry group.

coupled to n vector multiplets [13] has scalar fields whose interactions are governed by the coset $SO(n, 3)/SO(n) \times SO(3)$. The chiral reduction of this model gives rise to an $N = (1, 0)$ supergravity in $6D$ coupled to $p \leq n$ on-shell linear multiplets, one on-shell special linear multiplet and $(n - p)$ vector multiplets. The theory in $7D$ is half-maximal and coupled to n vector multiplets. Thus, it has $(40 + 8n)_B + (40 + 8n)_F$ degrees of freedom, while the chiral theory we end up with in $6D$ has $[16 + 4 + 4p + 4(n - p)]_B + [16 + 4 + 4p + 4(n - p)]_F$, that is to say $(20 + 4n)_B + (20 + 4n)_F$ degrees of freedom, which, as expected, is half the degrees of freedom we start with in $7D$.

An on-shell linear multiplet is the dual version of an off-shell linear multiplet whose bosons consist of a 4-form potential and three scalars, in which the 4-form potential is dualized to an axionic scalar. What we call an on-shell *special linear multiplet* has a bosonic sector that consists of three axionic scalars that can be dualized to three 4-form potentials and one scalar. We show that the model can be reformulated as a $6D$ supergravity theory coupled to $(p + 1)$ hypermultiplets whose scalar fields parametrize the coset $SO(p + 1, 4)/SO(p + 1) \times SO(4)$. This will be studied both in the symmetric gauge as well as in the Iwasawa gauge. The isometry group contains $(p + 3)$ abelian isometries which correspond to the $(n + 3)$ axions that can be dualized to the p linear and one special linear multiplets described above. This dualization is carried out here and the results are presented in Appendix C.

All of this is similar to the hypermultiplet coupling to $N = 2$ supergravity in $4D$ in which case the abelian isometries have a dual description in terms of a suitable number of tensor multiplets, which are the $4D$ version of our linear multiplets, and one double-tensor multiplet, which is the analog of our special linear multiplet (which we might view as triple-linear multiplet) [14].

Turning to the *gauged* half-maximal supergravities coupled to Yang-Mills in $7D$, the allowed semi-simple gauge groups are of the form

$$G_0 \times H \subset SO(n, 3) , \quad (1.1)$$

with G_0 being one of the six groups listed in (2.7). The noncompact gauge groups among them are those which have up to 3 compact or up to 3 noncompact generators. The models of special interest are those in which the chiral truncation of the $7D$ gauged theory gives rise to an R -symmetry gauged theory in $6D$. Such models are very difficult to obtain from higher dimensions, and indeed, only few such models exist. The ones we find are $Sp(1)_R$ or $U(1)_R$ gauged and matter coupled $6D$ supergravities with hidden $SO(p + 1, 4)/SO(p + 1) \times SO(4)$ structure for $p = 0, 1$, and one or no *external Maxwell multiplets* that do not participate in

the gauging of the R -symmetry. The content of these models will be summarized below. Analogous models have been constructed directly in $4D$ as $N = 2$ gauged supergravity coupled to hypermultiplets in which abelian isometries of the quaternionic Kahler scalar manifold are dualized to tensors and they are known to arise in certain compactifications of string theories [15, 16]².

As we shall see, the gauged chiral $6D$ supergravities arise from half-maximal $7D$ supergravities with *noncompact* gaugings. While noncompact gauging is necessary, it is not sufficient for obtaining R -symmetry gauging in $6D$. Indeed, of the five possible noncompact gaugings listed in (2.7), we shall find that the $SL(3, R)$ gauged theory does not admit a chiral circle reduction to a gauged $6D$ supergravity. In the case of $SO(2, 1)$ ³ gauged theory, it turns out that a consistent chiral reduction with surviving $6D$ gauge group $O(1, 1)$ ³ is possible but these do not gauge the R -symmetry. The remaining three noncompact gauged supergravities in $7D$, however, do give rise to the R -symmetry gauged supergravities in $6D$ and we find the following three models:

- **The $SO(3, 1)$ model:**

This model is obtained from the $SO(3, 1)$ gauged half-maximal $7D$ supergravity coupled to 3 vector multiplets, with $SO(3, 3)/SO(3) \times SO(3)$ scalar sector. Its chiral reduction gives a $U(1)_R$ *gauged supergravity coupled to a special linear multiplet in $6D$* , referred to as model II in section 3.2.

- **The $SO(2, 1)$ model:**

This model is obtained from the $SO(2, 1)$ gauged half-maximal $7D$ supergravity coupled to a single vector multiplet, with $SO(3, 1)/SO(3)$ scalar sector. Its chiral reduction gives rise to an $Sp(1)_R$ *gauged supergravity coupled to a special linear multiplet in $6D$* , referred to as model IV in section 3.2.

- **The $SO(2, 2)$ model:**

This model is obtained from the $SO(2, 2)$ gauged half-maximal $7D$ supergravity coupled to 3 vector multiplets, with $SO(3, 3)/SO(3)$ scalar sector. Its chiral reduction gives a $U(1)_R$ *gauged theory coupled to an additional Maxwell multiplet, a linear multiplet and a special linear multiplet in $6D$* , referred to as model V in section 3.2.

²In the context of globally supersymmetric sigma models, the phenomenon of dualizing abelian isometries of a hyperkahler manifold to obtain tensor multiplets in $4D$, and linear multiplets in $6D$, was described long ago in [17]. The $4D$ case was treated in more detail in [15] as well.

The $SO(2,2)$ and $SO(3,1)$ models can be obtained from a reduction of the $N = 1, D = 10$ supergravity on the noncompact hyperboloidal 3-manifolds $H_{2,2}$ and $H_{3,1}$, respectively [8, 9, 7]³. These models can also be obtained from analytical continuation of an $SO(4)$ gauged $7D$ supergravity [10] which, in turn, can be obtained from an S^3 compactification of Type IIA supergravity [11], or a limit of an S^4 reduction of $D = 11$ supergravity which reduces to a compactification on $S^3 \times R$ [12].

The specific gauged supergravities we have found are expected to play a role in a string/M theoretic construction of an anomaly-free matter coupled minimal gauged supergravity in $6D$. With regard to various matter coupled $7D$ supergravities considered here, we note that the heterotic string on T^3 gives rise to half-maximal $7D$ supergravity coupled to 19 Maxwell multiplets, which, in turn, is dual to M-theory on $K3$.

Finally, in this paper, we have also shown that, contrary to the claims made in the literature [18], we can dualize the 2-form potential to a 3-form potential in the gauged $7D$ supergravity even in the presence of couplings to an arbitrary number of vector multiplets (see Appendix A).

This paper is organized as follows. In section 2, we recall the (gauged) half-maximal supergravity couple to n vector multiplets [13]. In particular, we list the possible non-compact gaugings in this theory. In section 3, we determine the conditions that must be satisfied by the requirement of chiral supersymmetry in $6D$, both, for the gauged and ungauged $7D$ theory. We then solve these conditions, and determine the field content, the $6D$ supermultiplet structure. In section 4, we obtain the $6D$ supergravity for the fields that survive the chiral reduction, and their supersymmetry transformation rules. In section 5, we exhibit the hidden quaternionic Kahler coset structure that given the couplings of the matter multiplets in $6D$ by an extensive use of the Iwasawa decompositions. This is not surprising for the ungauged and bosonic sector. Here, we show the result for the gauged and ungauged models, and including the fermionic sectors well. A brief summary and comments are given in section 7.

In Appendix A, we give our result for the dual formulation of the gauged and matter coupled $7D$ supergravity, in which the 2-form potential is dualized to a 3-form potential. A useful relation between the $SL(4, R)/SO(4)$ parametrization used in [10, 7] and the $SO(3,3)/SO(3) \times SO(3)$ parametrization used in [13] and in this paper. In Appendix C, considering the ungauged $6D$ models, we dualize the axionic scalar field to 4-form potentials,

³These reductions can straightforwardly be lifted to $D = 11$. Note also that the spaces $H_{p,q}$ can be constructed from embedding into a (p, q) signature plane.

thereby obtaining a coupling of an arbitrary number of linear multiplets to a single special linear multiplet. Appendix D, contains some useful formula on the Iwasawa decomposition of $SO(p, q)$ that is used in showing the hidden quaternionic Kahler coset structure in six dimensional model. that we obtain by chiral reduction.

2 The Gauged 7D Model with Matter Couplings

Half-maximal supergravity in $D = 7$ coupled to n vector multiplets has the field content

$$\left(e_\mu{}^m, B_{\mu\nu}, \sigma, A_\mu^I, \phi^\alpha, \psi_\mu, \chi, \lambda^r \right) , \quad (2.1)$$

where the fermions $\psi_\mu, \chi, \lambda^r$ are symplectic Majorana and they all carry $Sp(1)$ doublet indices which have been suppressed. The $3n$ scalars $\phi^\alpha (\alpha = 1, 2, \dots, 3n)$ parametrize the coset

$$\frac{SO(n, 3)}{SO(n) \times SO(3)} . \quad (2.2)$$

The gauge fermions $\lambda^r (r = 1, \dots, n)$ transform in the vector representation of $SO(n)$, while the vector fields $A_\mu^I (I = 1, \dots, n + 3)$ transform in the vector representation of $SO(n, 3)$. The 2-form potential $B_{\mu\nu}$ and the dilaton σ are real.

It is useful to define a few ingredients associated with the scalar coset manifold as they arise in the Lagrangian. We first introducing the coset representative

$$L = (L_I^i, L_I^r) , \quad I = 1, \dots, n + 3, \quad i = 1, 2, 3 , \quad r = 1, \dots, n , \quad (2.3)$$

which forms an $(n + 3) \times (n + 3)$ matrix that obeys the relation

$$-L_I^i L_J^i + L_I^r L_J^r = \eta_{IJ} , \quad (2.4)$$

where $\eta_{IJ} = \text{diag}(- - - + + \dots)$. The contraction of the $SO(n)$ and $SO(3)$ indices is with the Kronecker deltas δ_{rs} and δ_{ij} while the raising and lowering of the $SO(n, 3)$ indices will be with the $SO(n, 3)$ invariant metric η_{IJ} . Given that the $SO(3)$ indices are raised and lowered by the Kronecker delta, it follows that, in our conventions,

$$L_I^i = L_{Ii} , \quad L_I^i L_J^I = -\delta_j^i , \quad L_I^i L^{Ij} = -\delta^{ij} .$$

Note also that the inverse coset representative L^{-1} is given by $L^{-1} = (L^I_i, L^I_r)$ where $L^I_i = \eta^{IJ} L_{Ji}$ and $L^I_r = \eta^{IJ} L_{Jr}$. In the gauged matter coupled theory of [13], a key

building block is the gauged Maurer-Cartan form

$$\begin{aligned}
P_\mu^{ir} &= L^{Ir} \left(\partial_\mu \delta_I^K + f_{IJ}^K A_\mu^J \right) L_K^i , \\
Q_\mu^{ij} &= L^{Ij} \left(\partial_\mu \delta_I^K + f_{IJ}^K A_\mu^J \right) L_K^i , \\
Q_\mu^{rs} &= L^{Ir} \left(\partial_\mu \delta_I^K + f_{IJ}^K A_\mu^J \right) L_K^s ,
\end{aligned} \tag{2.5}$$

where f_{IJ}^K are the structure constants of the not necessarily simple group $K \subset SO(n, 3)$ of dimension $n + 3$, and the gauge coupling constants are absorbed into their definition of the structure constants. The K -invariance of the theory requires that the adjoint representation of K leaves η_{IJ} invariant:

$$f_{IK}^L \eta_{LJ} + f_{JK}^L \eta_{LI} = 0 . \tag{2.6}$$

It follows that for each simple subgroup of K , the corresponding part of η_{IJ} must be a multiple of its Cartan-Killing metric. Since η_{IJ} contains an arbitrary number of positive entries, K can be an arbitrarily large compact group. On the other hand, as η_{IJ} has only three negative entries, K can have 3 or less compact generators, or 3 or less noncompact generators⁴. Thus, the allowed semi-simple gauge groups are of the form $G_0 \times H \subset SO(n, 3)$ where G_0 is one of the following

$$\begin{aligned}
(I) & \quad SO(3) \\
(II) & \quad SO(3, 1) \\
(III) & \quad SL(3, R) \\
(IV) & \quad SO(2, 1) \\
(V) & \quad SO(2, 1) \times SO(2, 1) \\
(VI) & \quad SO(2, 1) \times SO(2, 1) \times SO(2, 1)
\end{aligned} \tag{2.7}$$

and H is a semi-simple compact Lie group with $\dim H \leq (n + 3 - \dim G_0)$. Of these cases, only (I) with $H = SO(3)$ corresponding to $SO(4)$ gauged supergravity, (II) and (V) are known to have a ten- or eleven-dimensional origin. Though the cases (III)–(VI) are not mentioned explicitly in [13], the Lagrangian provided there is valid for all the cases listed above.

⁴This is similar to the reasoning in [19] where the gauging of $N = 4, D = 4$ supergravity coupled to n vector multiplets is considered. In this case, the relevant η is the $SO(n, 6)$ invariant tensor and the resulting noncompact simple gauge groups have been listed in [19].

The Lagrangian of [13], up to quartic fermion terms, is given by ⁵.

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F \quad (2.8)$$

$$\begin{aligned} e^{-1}\mathcal{L}_B &= \frac{1}{2}R - \frac{1}{4}e^\sigma \left(F_{\mu\nu}^i F_i^{\mu\nu} + F_{\mu\nu}^r F_r^{\mu\nu} \right) - \frac{1}{12}e^{2\sigma} G_{\mu\nu\rho} G^{\mu\nu\rho} \\ &\quad - \frac{5}{8}\partial_\mu\sigma\partial^\mu\sigma - \frac{1}{2}P_\mu^{ir} P_{ir}^\mu - \frac{1}{4}e^{-\sigma} \left(C^{ir} C_{ir} - \frac{1}{9}C^2 \right), \end{aligned} \quad (2.9)$$

$$\begin{aligned} e^{-1}\mathcal{L}_F &= -\frac{i}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{5i}{2}\bar{\chi}\gamma^\mu D_\mu\chi - \frac{i}{2}\bar{\lambda}^r\gamma^\mu D_\mu\lambda_r - \frac{5i}{4}\bar{\chi}\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu\sigma \\ &\quad - \frac{1}{2}\bar{\lambda}^r\sigma^i\gamma^\mu\gamma^\nu\psi_\mu P_{\nu ri} + \frac{i}{24\sqrt{2}}e^\sigma G_{\mu\nu\rho}X^{\mu\nu\rho} + \frac{1}{8}e^{\sigma/2}F_{\mu\nu}^i X_i^{\mu\nu} - \frac{i}{4}e^{\sigma/2}F_{\mu\nu}^r X_r^{\mu\nu} \\ &\quad - \frac{i\sqrt{2}}{24}e^{-\sigma/2}C \left(\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu + 2\bar{\psi}_\mu\gamma^\mu\chi + 3\bar{\chi}\chi - \bar{\lambda}^r\lambda_r \right) \\ &\quad + \frac{1}{2\sqrt{2}}e^{-\sigma/2}C_{ir} \left(\bar{\psi}_\mu\sigma^i\gamma^\mu\lambda^r - 2\bar{\chi}\sigma^i\lambda^r \right) + \frac{1}{2}e^{-\sigma/2}C_{rsi}\bar{\lambda}^r\sigma^i\lambda^s, \end{aligned} \quad (2.10)$$

where the fermionic bilinears are defined as

$$\begin{aligned} X^{\mu\nu\rho} &= \bar{\psi}^\lambda\gamma_{[\lambda}\gamma^{\mu\nu\rho}\gamma_{\tau]}\psi^\tau + 4\bar{\psi}_\lambda\gamma^{\mu\nu\rho}\gamma^\lambda\chi - 3\bar{\chi}\gamma^{\mu\nu\rho}\chi + \bar{\lambda}^r\gamma^{\mu\nu\rho}\lambda_r, \\ X_i^{\mu\nu} &= \bar{\psi}^\lambda\sigma^i\gamma_{[\lambda}\gamma^{\mu\nu}\gamma_{\tau]}\psi^\tau - 2\bar{\psi}_\lambda\sigma^i\gamma^{\mu\nu}\gamma^\lambda\chi + 3\bar{\chi}\sigma^i\gamma^{\mu\nu}\chi - \bar{\lambda}^r\sigma^i\gamma^{\mu\nu}\lambda_r, \\ X_r^{\mu\nu} &= \bar{\psi}_\lambda\gamma^{\mu\nu}\gamma^\lambda\lambda^r + 2\bar{\chi}\gamma^{\mu\nu}\lambda^r. \end{aligned} \quad (2.11)$$

The field strengths and the covariant derivatives are defined as

$$\begin{aligned} G_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]} - \frac{3}{\sqrt{2}}\omega_{\mu\nu\rho}^0, \quad \omega_{\mu\nu\rho}^0 = F_{[\mu\nu}^I A_{\rho]}^J \eta_{IJ} - \frac{1}{3}f_{IJ}^K A_\mu^I A_\nu^J A_{\rho K}, \\ F_{\mu\nu}^I &= 2\partial_{[\mu}A_{\nu]}^I + f_{JK}^I A_\mu^J A_\nu^K, \quad F_{\mu\nu}^i = F_{\mu\nu}^I L_I^i, \quad F_{\mu\nu}^r = F_{\mu\nu}^I L_I^r, \end{aligned} \quad (2.12)$$

$$D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} + \frac{1}{2\sqrt{2}}Q_\mu^i\sigma^i, \quad Q_\mu^i = \frac{i}{\sqrt{2}}\epsilon^{ijk}Q_{\mu jk}, \quad (2.13)$$

and the C -functions are given by [13]

$$\begin{aligned} C &= -\frac{1}{\sqrt{2}}f_{IJ}^K L_i^I L_j^J L_{Kk}\epsilon^{ijk}, \\ C_{ir} &= \frac{1}{\sqrt{2}}f_{IJ}^K L_j^I L_k^J L_{Kr}\epsilon^{ijk}, \\ C_{rsi} &= f_{IJ}^K L_r^I L_s^J L_{Ki}. \end{aligned} \quad (2.14)$$

⁵We follow the conventions of [13]. In particular, $\eta_{\mu\nu} = \text{diag}(- + + \dots +)$, the spinors are symplectic Majorana, $C^T = C$ and $(\gamma^\mu C)^T = -\gamma^\mu C$. Thus, $\bar{\psi}\gamma^{\nu_1\dots\nu_n}\lambda = (-1)^n\bar{\psi}\gamma^{\nu_n\dots\nu_1}\lambda$, where the $Sp(1)$ doublet indices are contracted and suppressed. Here we also use $X^A{}_B = \frac{1}{\sqrt{2}}(\sigma^i)^A{}_B X^i$, and further conventions are: $X^A = \epsilon^{AB}X_B$, $X_A = X^B\epsilon_{BA}$, $\epsilon^{AB}\epsilon_{BC} = -\delta_C^A$, $\bar{\psi}\lambda = \psi^A\lambda_A$, $\bar{\psi}\sigma^i\lambda = \bar{\psi}^A(\sigma^i)_A{}^B\epsilon_{B}$.

The local supersymmetry transformation rules read [13]

$$\begin{aligned}
\delta e_\mu^m &= i\bar{\epsilon}\gamma^m\psi_\mu, \\
\delta\psi_\mu &= 2D_\mu\epsilon - \frac{1}{60\sqrt{2}}e^\sigma G_{\rho\sigma\tau}(\gamma_\mu\gamma^{\rho\sigma\tau} + 5\gamma^{\rho\sigma\tau}\gamma_\mu)\epsilon \\
&\quad - \frac{i}{20}e^{\sigma/2}F_{\rho\sigma}^i\sigma^i(3\gamma_\mu\gamma^{\rho\sigma} - 5\gamma^{\rho\sigma}\gamma_\mu)\epsilon - \frac{\sqrt{2}}{30}e^{-\sigma/2}C\gamma_\mu\epsilon, \\
\delta\chi &= -\frac{1}{2}\gamma^\mu\partial_\mu\sigma\epsilon - \frac{i}{10}e^{\sigma/2}F_{\mu\nu}^i\sigma^i\gamma^{\mu\nu}\epsilon - \frac{1}{15\sqrt{2}}e^\sigma G_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon + \frac{\sqrt{2}}{30}e^{-\sigma/2}C\epsilon, \\
\delta B_{\mu\nu} &= i\sqrt{2}e^{-\sigma}(\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} + \bar{\epsilon}\gamma_{\mu\nu}\chi) - \sqrt{2}A_{[\mu}^I\delta A_{\nu]}^J\eta_{IJ}, \\
\delta\sigma &= -2i\bar{\epsilon}\chi, \\
\delta A_\mu^I &= -e^{-\sigma/2}(\bar{\epsilon}\sigma^i\psi_\mu + \bar{\epsilon}\sigma^i\gamma_\mu\chi)L_i^I + ie^{-\sigma/2}\bar{\epsilon}\gamma_\mu\lambda^r L_r^I, \\
\delta L_I^r &= \bar{\epsilon}\sigma^i\lambda^r L_I^i, \quad \delta L_I^i = \bar{\epsilon}\sigma^i\lambda_r L_I^r, \\
\delta\lambda^r &= -\frac{1}{2}e^{\sigma/2}F_{\mu\nu}^r\gamma^{\mu\nu}\epsilon + i\gamma^\mu P_\mu^{ir}\sigma^i\epsilon - \frac{i}{\sqrt{2}}e^{-\sigma/2}C^{ir}\sigma^i\epsilon.
\end{aligned} \tag{2.15}$$

For purposes of the next section, we exhibit the gauge field dependent part of the gauged Maurer-Cartan forms:

$$\begin{aligned}
P_\mu^{ir} &= P_\mu^{ir(0)} - \frac{1}{2\sqrt{2}}\epsilon^{ijk}C^{jr}A_\mu^k - C^{irs}A_\mu^s, \\
Q_\mu^{ij} &= Q_\mu^{ij(0)} + \frac{1}{3\sqrt{2}}\epsilon^{ijk}C A_\mu^k - \frac{1}{2\sqrt{2}}\epsilon^{ijk}C^{kr}A_\mu^r,
\end{aligned} \tag{2.16}$$

where the zero superscript indicates the gauge field independent parts.

3 Chiral Reduction on a Circle

3.1 Reduction Conditions

Here we shall consider all the $7D$ quantities of the previous section such as fields, world and Lorentz indices to be hatted, and the corresponding $6D$ quantities to be unhatted ones. We parametrize the $7D$ metric as

$$d\hat{s}^2 = e^{2\alpha\phi}ds^2 + e^{2\beta\phi}(dy - \mathcal{A})^2. \tag{3.1}$$

In order to obtain the canonical Hilbert-Einstein term in $D = 6$, we choose

$$\alpha = -\frac{1}{2\sqrt{10}}, \quad \beta = -4\alpha. \tag{3.2}$$

We shall work with the natural vielbein basis

$$\hat{e}^a = e^{\alpha\phi} e^a, \quad \hat{e}^7 = e^{\beta\phi} (dy - \mathcal{A}). \quad (3.3)$$

Next, we analyze the constraints that come from the requirement of circle reduction followed by chiral truncation retaining $N = (1, 0)$ supersymmetry. Let us first set to zero the $7D$ gauge coupling constant and deduce the consistent chiral truncation conditions. At the end of the section we shall then re-introduce the coupling constant and determine the additional constraints that need to be satisfied.

The gravitino field in seven dimensions splits into a left handed and a right handed gravitino in six dimensions upon reduction in a compact direction. Chiral truncation means that we set one of them to zero, say,

$$\hat{\psi}_{a-} = 0. \quad (3.4)$$

This condition, used in the supersymmetry variation of the vielbein, gravitino and the field $\hat{\chi}$ readily gives the following further conditions

$$\hat{F}_{ab}^{\hat{I}} L_{\hat{I}}^i = 0, \quad (3.5)$$

$$\mathcal{A}_a = 0, \quad \hat{G}_{ab7} = 0, \quad \hat{\psi}_{7+} = 0, \quad \hat{\chi}_+ = 0. \quad (3.6)$$

To see how we can satisfy the condition (3.5), it is useful to consider an explicit realization of the $SO(n, 3)/SO(n) \times SO(3)$ coset representative. A convenient such parametrization is given by

$$\hat{L} = \begin{pmatrix} \frac{1+\phi^t\phi}{1-\phi^t\phi} & \frac{2}{1-\phi^t\phi}\phi^t \\ \phi\frac{2}{1-\phi^t\phi} & 1 + \phi\frac{2}{1-\phi^t\phi}\phi^t \end{pmatrix} \quad (3.7)$$

where ϕ is a $n \times 3$ matrix $\phi_{\hat{r}i}$. Note that this is symmetric, and as such, we shall refer to this as the coset representative in the *symmetric gauge*. Now, we observe that to satisfy (3.5), we can split the index

$$\hat{I} = \{I, I'\}, \quad I = 1, \dots, p+3, \quad I' = p+4, \dots, n+3, \quad (3.8)$$

and set

$$\hat{A}_a^I = 0, \quad L_{I'}^i = 0. \quad (3.9)$$

Note that $0 \leq p \leq n$, and in particular, for $p = n$, all vector fields $A_\mu^I, I = 1, \dots, n+3$ vanish (i.e. there are no $A_\mu^{r'}$ fields) while all the coset scalars $\phi^{\hat{r}i}$ are nonvanishing⁶. For $p < n$,

⁶Note also that for $p = 0$, all coset scalars $\phi^{\hat{r}i}$ vanish while n vector field $A_\mu^{r'}$ survive.

however, as we shall see below, $(n - p)$ vector fields survive, and these, in turn, will play a role in obtaining a gauged supergravity in $6D$.

The second condition in (3.9) amounts to setting $\phi_{r'i} = 0$ and consequently, introducing the notation

$$\hat{r} = \{r, r'\} , \quad r = 1, \dots, p , \quad r' = p + 1, \dots, n , \quad (3.10)$$

we have

$$L_I{}^{r'} = 0 , \quad L_{I'}{}^r = 0 , \quad L_{I'}{}^{r'} = \delta_{I'}{}^{r'} . \quad (3.11)$$

Thus the surviving scalar fields are

$$\left(\hat{L}_I{}^i, \hat{L}_I{}^r \right) \equiv \left(L_I{}^i, L_I{}^r \right) , \quad I = 1, \dots, p + 3 , \quad i = 1, 2, 3 , \quad r = 1, \dots, p . \quad (3.12)$$

This is the coset representative of $SO(p, 3)/SO(p) \times SO(3)$. From the supersymmetric variations of the vanishing coset representatives $(L_{I'}{}^i, L_{I'}{}^{r'}, L_{I'}{}^r)$, on the other hand, we find that

$$\hat{\lambda}_+^r = 0 , \quad \hat{\lambda}_-^{r'} = 0 . \quad (3.13)$$

Using these results in the supersymmetry variation of $\hat{A}_7^{I'}$, in turn, immediately gives

$$\hat{A}_7^{I'} = 0 . \quad (3.14)$$

Next, defining

$$\hat{B} = B_{\mu\nu} dx^\mu \wedge dx^\nu + B_\mu dx^\mu \wedge dy , \quad (3.15)$$

the already found conditions $\hat{G}_{ab7} = \mathcal{A}_a = \hat{A}_a^I = 0$ imply that

$$B_\mu = 0 . \quad (3.16)$$

In summary, the surviving bosonic fields are

$$\left(g_{\mu\nu}, \phi, B_{\mu\nu}, \hat{\sigma}, \phi_{ir}, \hat{A}_7^I, A_\mu^{I'} \right) , \quad (3.17)$$

and the surviving fermionic fields are

$$\left(\hat{\psi}_{\mu+}, \hat{\psi}_{7-}, \hat{\chi}_-, \hat{\lambda}_-^r, \hat{\lambda}_+^{r'} \right) . \quad (3.18)$$

We will show in the next section that suitable combinations of these fields (see Eq. (4.1)) form the following supermultiplets:

$$(g_{\mu\nu}, B_{\mu\nu}, \sigma, \psi_\mu, \chi) , \quad (A_\mu^{I'}, \lambda^{r'}) , \quad (\phi_{ir}, \Phi^I, \varphi, \lambda^r, \psi) , \quad (3.19)$$

$$I = 1, \dots, p + 3 , \quad I' = p + 4, \dots, n + 3 ,$$

$$r = 1, \dots, p , \quad r' = p + 1, \dots, n , \quad i = 1, 2, 3 .$$

The last multiplet represents a fusion of p linear multiplets and one special linear multiplet, as explained in the introduction. In particular, the $(p+3)$ axionic scalars Φ^I can be dualized to 4-form potentials. Further truncations are possible. Setting $\phi_{ir} = 0$ gives one special linear multiplet with fields (Φ^i, φ, ψ) while setting $\Phi^I = 0$ eliminates all the (special) linear multiplets.

Extra Conditions due to Gauging

Extra conditions emerge upon turning on the 7D gauge coupling constants. They arise from the requirement that the gauge coupling constant dependent terms in the supersymmetry variations of $(\hat{\psi}_{a-}, \hat{\psi}_{7+}, \hat{\lambda}_+^r, \hat{\lambda}_-^{r'})$ vanish. These conditions are

$$\begin{aligned} C &= 0, & C^{ir} &= 0, \\ C^{irs} \Phi^s &= 0, & C^{ir's'} A_\mu^{s'} &= 0, \end{aligned} \quad (3.20)$$

where $\Phi^r = \Phi^I L_I^r$ and $A_\mu^{s'} = A_\mu^{I'} L_{I'}^{r'}$. More explicitly, these conditions take the form

$$\hat{f}_{IJK} L_i^I L_j^J L_k^K = 0, \quad \hat{f}_{IJK} L_i^I L_j^J L_r^K = 0, \quad (3.21)$$

$$\hat{f}_{IJK} L_i^I L_r^J \Phi^K = 0, \quad \hat{f}_{I'r's'} L_i^I A_\mu^{s'} = 0. \quad (3.22)$$

Solving these conditions, while keeping all $A_\mu^{r'}$ and Φ^I , results in a chiral gauged supergravity theory with the multiplets shown in (3.19) and gauge group $K' \subset SO(n, 3)$ with structure constants $\hat{f}_{r's't'}$. The scalars Φ^I transform in a $(p+3)$ dimensional representation of K' , and there are $3p$ scalars which parametrize the coset $SO(p, 3)/SO(p) \times SO(3)$. The nature of the R -symmetry gauge group can be read off from

$$D_\mu \epsilon = D_\mu^{(0)} \epsilon + \frac{1}{2\sqrt{2}} \sigma^i C^{ir'} A_\mu^{r'} \epsilon. \quad (3.23)$$

Note that the 6D model is R -symmetry gauged provided that $C^{ir'}$ does not vanish upon setting all scalars to zero. Moreover, an abelian R -symmetry group can arise when $\hat{f}_{r's't'}$ vanishes with $C^{kr'} \neq 0$. Next, we show how to solve the conditions (3.21) and (3.22).

3.2 Solution to the Reduction Conditions

The conditions (3.21) and (3.22) can be solved by setting

$$\hat{f}_{IJ}^K = 0, \quad \hat{f}_{I'J'}^K = 0. \quad (3.24)$$

Moreover, the structure constants of the $7D$ gauge group $G_0 \times H \subset SO(n, 3)$ must satisfy the condition (2.6):

$$\hat{f}_{\hat{I}\hat{K}}^{\hat{L}} \eta_{\hat{L}\hat{J}} + \hat{f}_{\hat{J}\hat{K}}^{\hat{L}} \eta_{\hat{L}\hat{I}} = 0 . \quad (3.25)$$

Given the $7D$ gauge groups listed in (2.7), we now check case by case when and how these conditions can be satisfied. To begin with, we observe that given the $G_0 \times H \subset SO(n, 3)$ gauged supergravity theory, the H sector can always be carried over to $6D$ dimension to give the corresponding Yang-Mills sector whose H -valued gauge fields do not participate in a possible R -symmetry gauging. Therefore, we shall consider the G_0 part of the $7D$ gauge group in what follows.

(I) $SO(3)$

In this model, the $7D$ gauge group is $SO(3)$ with structure constants

$$\hat{f}_{\hat{I}\hat{J}\hat{K}} = (g\epsilon_{IJK}, 0) . \quad (3.26)$$

To satisfy (3.24), we must set $g = 0$. Thus, we see that *a chiral truncation to a gauged $6D$ theory is not possible in this case.*

(II) $SO(3, 1)$

The smallest $7D$ scalar manifold that can accommodate this gauging is $SO(3, 3)/SO(3) \times SO(3)$. In the $7D$ theory, the gauge group $G_0 = SO(3, 1)$ can be embedded in $SO(3, 3)$ as follows. Denoting the $SO(3, 3)$ generators by $T_{AB} = (T_{ij}, T_{rs}, T_{ir})$, we can embed $SO(3, 1)$ by choosing the generators (T_{rs}, T_{3r}) which obey the commutation rules of the $SO(3, 1)$ algebra. These generators can be relabeled as

$$(T_{34}, T_{35}, T_{36}, T_{45}, T_{56}, T_{64}) = (T_1, T_2, T_3, T_4, T_5, T_6) \equiv (T_I, T_{I'}) , \quad (3.27)$$

with $I = 1, 2, 3$ and $I' = 4, 5, 6$. The algebra of these generators is given by

$$[T_I, T_J] = f_{IJ}^{K'} T_{K'} , \quad [T_{I'}, T_J] = f_{I'J}^K T_K , \quad [T_{I'}, T_{J'}] = f_{I'J'}^{K'} T_{K'} . \quad (3.28)$$

Thus, the conditions (3.24) are satisfied. Furthermore, the Cartan-Killing metric associated with this algebra is $(+++--)$ and it satisfies the condition (3.25). In this case, all the coset scalars are vanishing and the surviving matter scalar fields are (Φ^i, φ) which are the bosonic fields of a special linear multiplet. This sector will be shown to be described by the quaternionic Kahler coset $SO(4, 1)/SO(4)$ in section 5. We thus obtain an $Sp(1, R)$ *gauged*

supergravity in 6D coupled to a single hypermultiplet. In summary, we have the following chain of chiral circle reduction and hidden symmetry in this case:

$$\frac{SO(3,3)}{SO(3) \times SO(3)} \longrightarrow (\Phi^i, \varphi) \longrightarrow \frac{SO(4,1)}{SO(4)} \quad (3.29)$$

Note that the 7D theory we start with has $64_B + 64_F$ physical degrees of freedom, while the resulting 6D theory has $24_B + 24_F$ physical degrees of freedom.

(III) $SL(3, R)$

The minimal 7D scalar manifold to accommodate this gauging is $SO(5,3)/SO(5) \times SO(3)$. In the 7D theory, the gauge group is $SL(3, R)$, which has 3 compact and 5 noncompact generators. The condition (3.25) can be satisfied with $\eta = \text{diag}(- - - + + +)$ by making a particular choice of the generators of $SL(3, R)$ such as

$$(i\lambda_2, i\lambda_5, i\lambda_7, \lambda_1, \lambda_3, \lambda_4, \lambda_6, \lambda_8) = (T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8) = (T_I, T_{I'}) , \quad (3.30)$$

where $\lambda_1, \dots, \lambda_8$ are the standard Gell-Mann matrices, and $I = 1, \dots, p+3, I' = p+4, \dots, 8$ with $0 \leq p \leq 5$. However, the condition (3.24) is clearly not satisfied since $[T_1, T_2] = T_3$ and thus $\hat{f}_{IJ}^K \neq 0$. Therefore, we conclude that the *chiral truncation to a gauged 6D theory is not possible in this case.*

(IV) $SO(2, 1)$

For this gauging, the minimal 7D scalar manifold is $SO(3,1)/SO(3)$. Let us denote the generators of $SO(3,1)$ by $T_{AB} = (T_{ij}, T_{4i})$ where $i = 1, 2, 3$. The 7D gauge group $SO(2, 1)$ can be embedded into this $SO(3,1)$ by picking out the generators (T_{41}, T_{42}, T_{12}) , where the last generator is compact and the other two are noncompact. Thus,

$$\hat{f}_{\hat{I}\hat{J}\hat{K}} = (g \epsilon_{\underline{ijk}}, 0) , \quad \underline{i} = 1, 2, 4 , \quad (3.31)$$

where (T_{41}, T_{42}, T_{12}) correspond to (T_1, T_2, T_4) , respectively. The $SO(3,1)$ vector index, on the other hand, is labeled as $I = 1, 2, 3$ and $I' = 4$. Thus, the conditions (3.24) and (3.25) are satisfied and the resulting 6D theory is a $U(1)_R$ gauged supergravity coupled to one special linear multiplet. The gauge field is A_μ^4 , and the special linear multiplet lends itself to a description in terms of the quaternionic Kahler coset $SO(4,1)/SO(4)$. We thus obtain an $U(1)_R$ gauged supergravity in 6D coupled to one hypermultiplet. This model is similar to the $Sp(1)_R$ gauged model obtained from the $SO(3,1)$ gauged 7D supergravity described

above, the only difference being that the gauge group is now $U(1)_R$. In summary, we have the following chain of chiral circle reduction and hidden symmetry:

$$\frac{SO(3,1)}{SO(3)} \longrightarrow (\Phi^i, \varphi) \longrightarrow \frac{SO(4,1)}{SO(4)} \quad (3.32)$$

In this case, the 7D theory we start with has $64_B + 64_F$ physical degrees of freedom, while the resulting 6D theory has $24_B + 24_F$ physical degrees of freedom.

(V) $SO(2,2)$

This case is of considerable interest as it can be obtained from a reduction of $N = 1$ supergravity in ten dimensions on a certain manifold $H_{2,2}$ as shown in [7], where its chiral circle reduction has been studied. As we shall see below, their result is a special case of a more general such reduction.

The minimal model that can accommodate the $SO(2,2)$ gauging is $SO(3,3)/SO(3) \times SO(3) \sim SL(4, R)/SO(4)$. To solve the conditions (3.24), we embed the $SO(2,2)$ in $SO(3,3)$ by setting

$$\begin{aligned} \hat{f}_{\hat{I}\hat{J}}^{\hat{K}} &= (g_1 \epsilon_{\underline{ij}\ell} \eta^{\underline{k}\ell}, g_2 \epsilon_{\underline{rst}} \eta^{\underline{tq}}), \quad \underline{i} = 1, 2, 6, \quad \underline{r} = 3, 4, 5, \\ \eta_{\underline{ij}} &= \text{diag}(- - +), \quad \eta_{\underline{rs}} = \text{diag}(- + +), \end{aligned} \quad (3.33)$$

where (g_1, g_2) are the gauge coupling constants for $SO(2,1) \times SO(2,1) \sim SO(2,2)$. These structure constants can be checked to satisfy the condition (3.25). Furthermore, the conditions (3.24) are satisfied since $I = 1, 2, 3, 4$ and $I' = 5, 6$. The resulting 6D theory is *a $U(1)_R$ gauged supergravity coupled to one external Maxwell multiplet (in addition to the Maxwell multiplet that gauges the R-symmetry) and two hypermultiplets*. The two hypermultiplets consist of the fields shown in the last group in (3.19) with $p = 1, n = 3$. The $U(1)_R$ is gauged by the vector field A_μ^6 . The vector field A_μ^5 , which corresponds to $O(1,1)$ rotations, resides in the Maxwell multiplet. In this model, the surviving $SO(3,1)/SO(3)$ sigma model sector in 6D, gets enlarged with the help of the axionic fields to become the quaternionic Kahler coset $SO(4,2)/SO(4) \times SO(2)$, as will be shown in section 5. In summary, we have the following chain of chiral circle reduction and hidden symmetry

$$\frac{SO(3,3)}{SO(3) \times SO(3)} \longrightarrow \frac{SO(3,1)}{SO(3)} \longrightarrow \frac{SO(4,2)}{SO(4) \times SO(2)} \quad (3.34)$$

It is also worth noting that the Cvetic-Gibbons-Pope reduction [7] that gave rise to the $U(1)_R$ gauged 6D supergravity is a special case of our results that can be obtained by setting to zero all the scalar fields of the $SO(3,1)/SO(3)$ sigma model, the gauge field A_μ^5 and their

fermionic partners. This model was studied in the language of the $SL(4, R)/SO(4)$ coset structure. In Appendix B, we give the map between this coset and the $SO(3, 3)/SO(3) \times SO(3)$ coset used here.

Note that, in this case the $7D$ theory we start with has $64_B + 64_F$ physical degrees of freedom, and the resulting $6D$ theory has half as many, namely, $32_B + 32_F$ physical degrees of freedom

(VI) $SO(2, 2) \times SO(2, 1)$

In this case, the minimal $7D$ sigma model sector is based on $SO(6, 3)/SO(6) \times SO(3)$. To solve the condition (3.24) in such a way to obtain an R -symmetry gauged $6D$ supergravity, we embed the $7D$ gauge group $SO(2, 2) \times SO(2, 1)$ in $SO(6, 3)$ by setting

$$\begin{aligned} \hat{f}_{\hat{I}\hat{J}}^{\hat{K}} &= \left(g_1 \epsilon_{\underline{ij}\ell} \eta^{\underline{k}\ell}, g_2 \epsilon_{\underline{rst}} \eta^{\underline{tq}}, g_3 \epsilon_{i'j'\ell'} \eta^{k'\ell'} \right), \quad \underline{i} = 1, 4, 5, \quad \underline{r} = 2, 6, 7, \quad i' = 3, 8, 9, \\ \eta_{\underline{ij}} &= \text{diag}(-++) , \quad \eta_{\underline{rs}} = \text{diag}(-++) , \quad \eta_{i'j'} = \text{diag}(-++) , \end{aligned} \quad (3.35)$$

where (g_1, g_2, g_3) are the gauge coupling constants for $SO(2, 1) \times SO(2, 1) \times SO(2, 1)$. The conditions (3.24) are satisfied since $I = 1, 2, 3, 5, 7, 9$ and $I' = 4, 6, 8$. *The resulting $6D$ theory has a local $O(1, 1)^3$ gauge symmetry, and hence three Maxwell multiplets but no gauged R symmetry, and three hypermultiplets.* The gauge fields are $(A_\mu^4, A_\mu^6, A_\mu^8)$, and the hypermultiplets consist of the fields shown in the last group in (3.19) with $p = 3, n = 6$. In this model, the surviving $SO(3, 3)/SO(3) \times SO(3)$ sigma model in $6D$ gets enlarged to the quaternionic Kahler $SO(4, 4)/SO(4) \times SO(4)$ with the help of the axionic fields, as will be described in section 5. In summary, we have the following chain of chiral circle reduction and hidden symmetry:

$$\frac{SO(6, 3)}{SO(6) \times SO(3)} \longrightarrow \frac{SO(3, 3)}{SO(3) \times SO(3)} \longrightarrow \frac{SO(4, 4)}{SO(4) \times SO(4)} \quad (3.36)$$

To summarize, we have found that the $SO(3, 1)$ and $SO(2, 2)$ gauged $7D$ models give rise to $U(1)_R$ gauged supergravity, and the $SO(2, 1)$ gauged $7D$ model yields an $Sp(1)_R$ gauged chiral supergravity, coupled to specific matter multiplets in six dimensions.

4 The $6D$ Lagrangian and Supersymmetry Transformations

The chiral reduction on a circle along the lines described above requires, as usual, the diagonalization of the kinetic terms for various matter fields. This is achieved by defining

$$\begin{aligned}
\sigma &= (\hat{\sigma} - 2\alpha\phi) , & \varphi &= \frac{1}{2}(\hat{\sigma} + 8\alpha\phi) , \\
\chi &= \sqrt{2}e^{\alpha\phi/2} \left(\hat{\chi} + \frac{1}{4}\hat{\psi}_7 \right) , & \psi &= \frac{1}{\sqrt{2}}e^{\alpha\phi/2} (\hat{\psi}_7 - \hat{\chi}) , \\
\psi_a &= \frac{1}{\sqrt{2}}e^{\alpha\phi/2} \left(\hat{\psi}_a - \frac{1}{4}\gamma_a\hat{\psi}_7 \right) , & \psi^r &= \frac{1}{\sqrt{2}}e^{\alpha\phi/2}\hat{\lambda}^r , \\
\Phi^I &= \hat{A}_7^I , & \lambda^{r'} &= \frac{1}{\sqrt{2}}e^{\alpha\phi/2}\hat{\lambda}^{r'} , & \hat{\epsilon} &= \frac{1}{\sqrt{2}}e^{\alpha\phi/2}\epsilon .
\end{aligned} \tag{4.1}$$

Furthermore, recalling the Ansatz and noting that the only non-vanishing components of the spin connection are

$$\hat{\omega}_{cab} = e^{-\alpha\phi} \left(\omega_{cab} + 2\alpha\eta_{c[a}\partial_{b]}\phi \right) , \quad \hat{\omega}_{77a} = \beta e^{-\alpha\phi} \partial_a\phi , \tag{4.2}$$

where the indices on the spin connection refer to the tangent space, the 6D supergravity theory obtained by the reduction scheme describe above has the Lagrangian $\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F$ where⁷

$$\begin{aligned}
e^{-1}\mathcal{L}_B &= \frac{1}{4}R - \frac{1}{4}(\partial_\mu\sigma)^2 - \frac{1}{12}e^{2\sigma}G_{\mu\nu\rho}G^{\mu\nu\rho} - \frac{1}{8}e^\sigma F_{\mu\nu}^{r'}F^{\mu\nu r'} \\
&\quad - \frac{1}{4}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{4}P_\mu^{ir}P_{ir}^\mu - \frac{1}{4}\left(P_\mu^rP_r^\mu + \mathcal{P}_\mu^i\mathcal{P}_i^\mu\right) \\
&\quad - \frac{1}{8}e^{-\sigma}\left(C^{ir'}C_{ir'} + 2S^{ir'}S_{ir'}\right) ,
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
e^{-1}\mathcal{L}_F &= -\frac{i}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{i}{2}\bar{\chi}\gamma^\mu D_\mu\chi - \frac{i}{2}\bar{\lambda}^{r'}\gamma^\mu D_\mu\lambda_{r'} \\
&\quad - \frac{i}{2}\bar{\psi}\gamma^\mu D_\mu\psi - \frac{i}{2}\bar{\psi}^r\gamma^\mu D_\mu\psi^r - \frac{i}{2}\bar{\chi}\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu\sigma \\
&\quad - \frac{1}{2}\bar{\psi}^r\gamma^\mu\gamma^\nu\sigma_i\psi_\mu P_\nu^{ir} + \frac{i}{2}\bar{\psi}\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu\varphi \\
&\quad - \frac{1}{2}\bar{\psi}\gamma^\mu\gamma^\nu\sigma_i\psi_\mu\mathcal{P}_\nu^i - \frac{i}{2}\bar{\psi}^r\gamma^\mu\gamma^\nu\psi_\mu P_\nu^r - \frac{1}{4}\mathcal{P}_\mu^i X_i^\mu \\
&\quad - iP_\mu^r X_r^\mu + \frac{i}{24}e^\sigma G_{\mu\nu\rho}X^{\mu\nu\rho} - \frac{i}{4}e^{\sigma/2}F_{\mu\nu}^{r'}X_{r'}^{\mu\nu} \\
&\quad + e^{-\sigma/2}\left(-C_{irr'}\bar{\lambda}^{r'}\sigma^i\psi^r + iS_{rr'}\bar{\lambda}^{r'}\psi^r - S_{ir'}\bar{\lambda}^{r'}\sigma^i\psi\right) \\
&\quad + \frac{1}{2\sqrt{2}}e^{-\sigma/2}\bar{\lambda}^{r'}\sigma^i\gamma^\mu\psi_\mu\left(C_{ir'} - \sqrt{2}S_{ir'}\right) \\
&\quad + \frac{1}{2\sqrt{2}}e^{-\sigma/2}\bar{\lambda}^{r'}\sigma^i\chi\left(C_{ir'} - \sqrt{2}S_{ir'}\right) ,
\end{aligned} \tag{4.4}$$

⁷In order to make contact with more standard conventions in 6D, we have redefined $G_{\mu\nu\rho} \rightarrow \sqrt{2}G_{\mu\nu\rho}$ and multiplied the Lagrangian by a factor of 1/2. The spacetime signature is $(-++++)$, the spinors are symplectic Majorana-Weyl, $C^T = -C$ and $(\gamma^\mu C)^T = -\gamma^\mu C$. Thus, $\bar{\psi}\gamma^{\nu_1\cdots\nu_n}\lambda = (-1)^n\bar{\psi}\gamma^{\nu_n\cdots\nu_1}\lambda$, where the $Sp(1)$ doublet indices are contracted and suppressed. We also use the convention: $\gamma_{\mu_1\cdots\mu_6} = \epsilon\epsilon_{\mu_1\cdots\mu_6}\gamma_7$.

and where

$$\begin{aligned}
X^{\mu\nu\rho} &= \bar{\psi}^\lambda \gamma_{[\lambda} \gamma^{\mu\nu\rho} \gamma_{\tau]} \psi^\tau + \bar{\psi}_\lambda \gamma^{\mu\nu\rho} \gamma^\lambda \chi - \bar{\chi} \gamma^{\mu\nu\rho} \chi + \bar{\lambda}^{r'} \gamma^{\mu\nu\rho} \lambda_{r'} + \bar{\psi}^r \gamma^{\mu\nu\rho} \psi_r + \bar{\psi} \gamma^{\mu\nu\rho} \psi , \\
X_i^\mu &= \bar{\psi}^\rho \gamma_{[\rho} \gamma^\mu \gamma_{\tau]} \sigma_i \psi^\tau + \bar{\chi} \gamma^\mu \sigma_i \chi + \bar{\lambda}^{r'} \gamma^\mu \sigma_i \lambda_{r'} - \bar{\psi}^r \gamma^\mu \sigma_i \psi_r - \bar{\psi} \gamma^\mu \sigma_i \psi , \\
X_{r'}^{\mu\nu} &= \bar{\psi}_\rho \gamma^{\mu\nu} \gamma^\rho \lambda_{r'} + \bar{\chi} \gamma^{\mu\nu} \lambda_{r'} , \\
X_r^\mu &= \bar{\psi} \gamma^\mu \psi_r .
\end{aligned} \tag{4.5}$$

The action is invariant under the following 6D supersymmetry transformations

$$\begin{aligned}
\delta e_\mu^m &= i \bar{\epsilon} \gamma^m \psi_\mu , \\
\delta \psi_\mu &= D_\mu \epsilon - \frac{1}{24} e^\sigma \gamma^{\rho\sigma\tau} \gamma_\mu G_{\rho\sigma\tau} \epsilon - \frac{i}{2} \mathcal{P}_\mu^i \sigma^i \epsilon , \\
\delta \chi &= -\frac{1}{2} \gamma^\mu \partial_\mu \sigma \epsilon - \frac{1}{12} e^\sigma \gamma^{\rho\sigma\tau} G_{\rho\sigma\tau} \epsilon , \\
\delta B_{\mu\nu} &= i e^{-\sigma} \left(\bar{\epsilon} \gamma_{[\mu} \psi_{\nu]} + \frac{1}{2} \bar{\epsilon} \gamma_{\mu\nu} \chi \right) - A_{[\mu}^{r'} \delta A_{\nu]}^{r'} , \\
\delta \sigma &= -i \bar{\epsilon} \chi , \\
\delta A_\mu^{r'} &= i e^{-\sigma/2} \bar{\epsilon} \gamma_\mu \lambda^{r'} , \\
\delta \lambda^{r'} &= -\frac{1}{4} e^{\sigma/2} \gamma^{\mu\nu} F_{\mu\nu}^{r'} \epsilon - \frac{i}{2\sqrt{2}} e^{-\sigma/2} \left(C^{ir'} - \sqrt{2} S^{ir'} \right) \sigma^i \epsilon , \\
L_I^r \delta \phi^I &= -i e^{-\varphi} \bar{\epsilon} \psi^r , \\
L_I^i \delta \phi^I &= e^{-\varphi} \bar{\epsilon} \sigma^i \psi , \\
L_i^I \delta L_I^r &= -\bar{\epsilon} \sigma_i \psi^r , \\
\delta \varphi &= i \bar{\epsilon} \psi , \\
\delta \psi &= \frac{i}{2} \gamma^\mu \left(\mathcal{P}_\mu^i \sigma_i - i \partial_\mu \varphi \right) \epsilon , \\
\delta \psi^r &= \frac{i}{2} \gamma^\mu \left(P_\mu^{ir} \sigma_i + i P_\mu^r \right) \epsilon .
\end{aligned} \tag{4.6}$$

Several definitions are in order. Firstly, the gauged Maurer-Cartan form is associated with the coset $SO(p,3)/SO(p) \times SO(3)$ and it is defined as

$$\begin{aligned}
P_\mu^{ir} &= L^{Ii} \left(\partial_\mu \delta_I^J - f_{r'I}^J A_\mu^{r'} \right) L_J^r , \\
Q_\mu^{ij} &= L^{Ii} \left(\partial_\mu \delta_I^J - f_{r'I}^J A_\mu^{r'} \right) L_J^j , \\
Q_\mu^{rs} &= L^{Ir} \left(\partial_\mu \delta_I^K - f_{r'I}^J A_\mu^{r'} \right) L_J^s .
\end{aligned} \tag{4.7}$$

Various quantities occurring above are defined as follows:

$$\begin{aligned}
G_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]} - \frac{3}{2}\left(F_{[\mu\nu]}^{r'}A_{\rho]}^{r'} - \frac{1}{3}f_{r's't'}A_{\mu}^{r'}A_{\nu}^{s'}A_{\rho t'}\right) , \\
F_{\mu\nu}^{r'} &= 2\partial_{[\mu}A_{\nu]}^{r'} + f_{s't'}{}^{r'}A_{\mu}^{s'}A_{\nu}^{t'} , \\
D_{\mu}\epsilon &= \left(\partial_{\mu} + \frac{1}{4}\omega_{\mu}{}^{ab}\gamma_{ab} + \frac{1}{2\sqrt{2}}Q_{\mu}^i\sigma^i\right)\epsilon , \\
Q_{\mu}^i &= \frac{i}{\sqrt{2}}\epsilon^{ijk}Q_{\mu jk} = \epsilon^{ijk}(L^{-1}\partial_{\mu}L)_{jk} + C^{ir'}A_{\mu}^{r'} .
\end{aligned} \tag{4.8}$$

The axion field strengths are defined as

$$\begin{aligned}
\mathcal{P}_{\mu}^i &= e^{\varphi}(D_{\mu}\phi^I)L_I^i , \\
P_{\mu}^r &= e^{\varphi}(D_{\mu}\phi^I)L_I^r , \\
D_{\mu}\Phi^I &= \partial_{\mu}\Phi^I + f_{r'I}{}^J A_{\mu}^{r'}\Phi^J ,
\end{aligned} \tag{4.9}$$

and the gauge functions as

$$\begin{aligned}
C_{kr'} &= \frac{1}{\sqrt{2}}\epsilon_{kij}f_{r'I}{}^J L^I{}_i L_J^j , & C_{irr'} &= f_{r'I}{}^J L_i^I L_{Jr} , \\
S_{ir'} &= -e^{\varphi}f_{r'I}{}^J \Phi_J L_i^I , & S_{rr'} &= -e^{\varphi}f_{r'I}{}^J \Phi_J L_r^I .
\end{aligned} \tag{4.10}$$

Note that the S^2 term in the potential in (4.3) comes from the $P_7^{ir'}P_7^{ir'}$ term since $P_7^{ir'} \sim S^{ir'}$. The above results cover all the chiral reduction schemes that yield gauged supergravities in $6D$. We simply need to take the appropriate structure constants and the relevant values of p in the $SO(p,3)/SO(p) \times SO(3)$ cosets involved.

In the case of the models (II) and (IV), with $7D$ gauge groups $SO(3,1)$ and $SO(2,1)$, respectively, we have $p = 0$, which means that the coset representative becomes an identity matrix and

$$\begin{aligned}
f_{rIJ} &\rightarrow \epsilon_{r'ij} , & C_{ijr'} &\rightarrow \sqrt{2}\epsilon_{r'ij} , & C_{irr'} &\rightarrow 0 , \\
S_{ir'} &\rightarrow -\epsilon_{r'ij}e^{\varphi}\Phi^j , & S_{rr'} &\rightarrow 0 .
\end{aligned} \tag{4.11}$$

By an untwisting procedure, which will be described in the next section, the scalar fields (Φ^i, φ) can be combined to describe the quaternionic Kahler manifold $SO(4,1)/SO(4)$ that governs the couplings of a single hypermultiplet.

In the case of Model (V), we have $p = 1$, which means that in the $6D$ model presented above, the relevant sigma model is $SO(3,1)/SO(3)$, while in the case of Model (VI), we have $p = 2$, which implies the sigma model $SO(4,2)/SO(4) \times SO(2)$. In each case, the range of indices I, r, i are fixed accordingly.

5 The Hidden Quaternionic Kahler Coset Structure

It is well known that the ten dimensional $N = 1$ supergravity theory coupled to N Maxwell multiplets when reduced on a k -dimensional torus down to D dimensions gives rise to half-maximal supergravity coupled to $(N + k)$ vector multiplets with an underlying $SO(N + k, k)/SO(N + k) \times SO(k)$ sigma model sector. This means an $SO(N + 3, 3)/SO(N + 3) \times SO(3)$ sigma model in $7D$. In the notation of the previous sections, we have $N + 3 = n$. A circle reduction of this ungauged theory is then expected to exhibit an $SO(N + 4, 4)/SO(N + 4) \times SO(4)$ coset structure. This is a well known phenomenon which has been described in several papers but primarily in the bosonic sector. In this section, we shall exhibit this phenomenon in the fermionic sector as well, including the supersymmetry transformations. Moreover, we shall describe the hidden symmetry of the gauged $6D$ models obtained from a consistent chiral reduction of the gauged $7D$ models, in which case the $SO(p + 3, 3)/SO(p + 3) \times SO(3)$ coset is enlarged to $SO(p + 4, 4)/SO(p + 4) \times SO(4)$. Here, we have redefined $p \rightarrow p + 3$ compared to the notation of the previous section, for convenience.

The key step in uncovering the hidden symmetry is to first rewrite the Lagrangian in Iwasawa gauge. This gauge is employed by parametrizing the coset $SO(p + 3, 3)/SO(p + 3) \times SO(3) \equiv C(p + 3, 3)$ by means of the $3(p + 3)$ dimensional solvable subalgebra K_s of $SO(p + 3, 3)$. The importance of this gauge lies in the fact that it enables one to absorb the $(p + 6)$ axions that come from the $7D$ Maxwell fields, and a single dilaton that comes from the $7D$ metric, into the representative of the coset $C(p + 3, 3)$ to form the representative of the enlarged coset $C(p + 4, 4)$ ⁸. To do so, we shall first show, in section 5.1, how various quantities *formally* combine to give the enlarged coset structure. This will involve identifications such as those in (5.1) below. These identifications by themselves do not furnish a proof of the enlarged coset structure, since one still has to construct explicitly a parametrization of the enlarged coset which produces these identifications. In section 5.2, we shall provide the proof by exploiting the Iwasawa gauge.

⁸In general, the solvable subalgebra $\widehat{K}_s \subset SO(p + k + 1, k + 1)$ decomposes into the generators $K_s \subset SO(p + k, k)$, and $(p + 2k)$ generators corresponding to axions and a single generator corresponding to a dilaton.

5.1 Hidden Symmetry in the Symmetric Gauge

The structure of the Lagrangian and transformation rules presented above readily suggest the identifications $\hat{P}^{ir} = P^{ir}$, $\hat{Q}^{ij} = Q^{ij}$, $\hat{Q}^{rs} = Q^{rs}$ and

$$\begin{pmatrix} \hat{P}^{4r} \\ \hat{P}^{i,N+4} \\ \hat{P}^{4,N+4} \end{pmatrix} = \begin{pmatrix} P^r \\ \mathcal{P}^i \\ -\partial\varphi \end{pmatrix}, \quad \begin{pmatrix} \hat{Q}^{4i} \\ \hat{Q}^{N+4,r} \end{pmatrix} = \begin{pmatrix} \mathcal{P}^i \\ P^r \end{pmatrix} \quad (5.1)$$

for the components of the Maurer-Cartan form, $\hat{C}^{ijr'} = C^{ijr'}$, $\hat{C}^{irr'} = C^{irr'}$ and the following identifications

$$\begin{pmatrix} \hat{C}^{4ir'} \\ \hat{C}^{4rr'} \\ \hat{C}^{i,N+4,r'} \\ \hat{C}^{4,N+4,r'} \end{pmatrix} = \begin{pmatrix} S^{ir'} \\ S^{rr'} \\ -S^{ir'} \\ 0 \end{pmatrix}, \quad (5.2)$$

where $\hat{C}^{ijr'} = \frac{1}{\sqrt{2}} \epsilon^{ijk} C^{kr'}$, for the gauge functions. Note that the hat notation here does not refer to higher dimensions but rather they denote objects which transform under the enlarged symmetry groups. With these identifications the Lagrangian simplifies dramatically. The bosonic part takes the form

$$\begin{aligned} e^{-1} \mathcal{L}_B &= \frac{1}{4} R - \frac{1}{4} (\partial_\mu \sigma)^2 - \frac{1}{12} e^{2\sigma} G_{\mu\nu\rho} G^{\mu\nu\rho} - \frac{1}{8} e^\sigma F_{\mu\nu}^{r'} F^{\mu\nu r'} \\ &\quad - \frac{1}{4} \hat{P}_\mu^{\hat{i}\hat{r}} \hat{P}_\mu^{\hat{i}\hat{r}} - \frac{1}{8} e^{-\sigma} \hat{C}^{\hat{i}\hat{j}r'} \hat{C}_{\hat{i}\hat{j}r'}, \end{aligned} \quad (5.3)$$

and the fermionic part is given by

$$e^{-1} \mathcal{L}_F = -\frac{i}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{i}{2} \bar{\chi} \gamma^\mu D_\mu \chi - \frac{i}{2} \bar{\lambda}^{r'} \gamma^\mu D_\mu \lambda_{r'} - \frac{i}{2} \bar{\psi}^{\hat{r}} \gamma^\mu D_\mu \psi^{\hat{r}} \quad (5.4)$$

$$\begin{aligned} & -\frac{i}{2} \bar{\chi} \gamma^\mu \gamma^\nu \psi_\mu \partial_\nu \sigma - \frac{1}{2} \bar{\psi}^{\hat{r}} \gamma^\mu \gamma^\nu \bar{\Gamma}_{\hat{i}} \psi_\mu \hat{P}_\nu^{\hat{i}\hat{r}} + \frac{i}{24} e^\sigma G_{\mu\nu\rho} X^{\mu\nu\rho} \\ & -\frac{i}{4} e^{\sigma/2} F_{\mu\nu}^{r'} X_{r'}^{\mu\nu} - e^{-\sigma/2} \hat{C}_{\hat{i}\hat{r}r'} \bar{\lambda}^{r'} \Gamma_{\hat{i}} \psi^{\hat{r}} \\ & -\frac{i}{4} e^{-\sigma/2} \hat{C}^{\hat{i}\hat{j}r'} \left(\bar{\lambda}^{r'} \Gamma_{\hat{i}\hat{j}} \gamma^\mu \psi_\mu + \bar{\lambda}^{r'} \Gamma_{\hat{i}\hat{j}} \chi \right), \end{aligned} \quad (5.5)$$

where $\hat{i} = 1, \dots, 4$, $\hat{r} = 1, \dots, p+4$, we have defined $\psi^{N+4} = \psi$, and

$$\begin{aligned} X^{\mu\nu\rho} &= \bar{\psi}^\lambda \gamma_{[\lambda} \gamma^{\mu\nu\rho} \gamma_{\tau]} \psi^\tau + \bar{\psi}_\lambda \gamma^{\mu\nu\rho} \gamma^\lambda \chi - \bar{\chi} \gamma^{\mu\nu\rho} \chi + \bar{\lambda}^{r'} \gamma^{\mu\nu\rho} \lambda_{r'} + \bar{\psi}^{\hat{r}} \gamma^{\mu\nu\rho} \psi^{\hat{r}}, \\ X_{r'}^{\mu\nu} &= \bar{\psi}_\rho \gamma^{\mu\nu} \gamma^\rho \lambda_{r'} + \bar{\chi} \gamma^{\mu\nu} \lambda_{r'}. \end{aligned} \quad (5.6)$$

The covariant derivatives are defined as

$$D_\mu \begin{pmatrix} \psi_\mu \\ \chi \\ \lambda^{r'} \end{pmatrix} = \left(\nabla_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} + \frac{1}{4} \hat{Q}_\mu^{\hat{i}\hat{j}} \Gamma_{\hat{i}\hat{j}} \right) \begin{pmatrix} \psi_\mu \\ \chi \\ \lambda^{r'} \end{pmatrix},$$

$$D_\mu \psi^{\hat{r}} = \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} + \frac{1}{4} \hat{Q}_\mu^{\hat{i}\hat{j}} \bar{\Gamma}_{\hat{i}\hat{j}} \right) \psi^{\hat{r}} + \hat{Q}_\mu^{\hat{r}\hat{s}} \psi^{\hat{s}} . \quad (5.7)$$

The $SO(4)$ Dirac matrices have been introduced in the above formula with the conventions

$$\Gamma_{\hat{i}} = (\sigma_i, -i) , \quad \bar{\Gamma}_{\hat{i}} = (\sigma_i, i) , \quad \Gamma_{\hat{i}\hat{j}} = \Gamma_{[\hat{i}} \bar{\Gamma}_{\hat{j}]} , \quad \bar{\Gamma}_{\hat{i}\hat{j}} = \bar{\Gamma}_{[\hat{i}} \Gamma_{\hat{j}]} . \quad (5.8)$$

It is useful to note that $\bar{\psi}^{\hat{r}} \Gamma^{\hat{i}} \epsilon = -\bar{\epsilon} \Gamma^{\hat{i}} \psi^{\hat{r}}$. Looking more closely at the covariant derivatives

$$\begin{aligned} D_\mu \chi &= \left(D_\mu(\omega) + \frac{1}{4} Q_\mu^{ij} \sigma_{ij} + \frac{i}{2} \mathcal{P}_\mu^i \sigma_i \right) \chi , \\ D_\mu \psi^{\hat{r}} &= \left(D_\mu(\omega) + \frac{1}{4} Q_\mu^{ij} \sigma_{ij} - \frac{i}{2} \mathcal{P}_\mu^i \sigma_i \right) \psi^{\hat{r}} + \hat{Q}_\mu^{\hat{r}\hat{s}} \psi^{\hat{s}} , \end{aligned} \quad (5.9)$$

we observe that they transform covariantly under the composite local $Sp(1)_R$ transformations inherited from $7D$, and that they contain the composite $Sp(1)_R$ connections shifted by the *positive torsion* term ($\frac{i}{2} \mathcal{P}^i \sigma^i$) in the case of fermions that are doublets under the true $Sp(1)_R$ symmetry group in $6D$, namely $(\psi_\mu, \chi, \lambda^{r'})$, and the *negative torsion* term ($-\frac{i}{2} \mathcal{P}^i \sigma^i$) in the case of $\psi^{\hat{r}}$, which are singlets under this symmetry. By true $Sp(1)_R$ symmetry group in $6D$ we mean the $SO(3)_R$ symmetry group that emerges upon the recognition of the scalar field couplings as being described by the quaternionic Kahler coset $SO(p+1, 4)/SO(p+1) \times SO(4)$ in which $SO(4) \sim SO(3) \times SO(3)_R$. The action of this group is best seen by employing the Iwasawa gauge, as we shall see in the next subsection. The action is invariant under the following $6D$ supersymmetry transformations

$$\begin{aligned} \delta e_\mu^m &= i \bar{\epsilon} \gamma^m \psi_\mu , \\ \delta \psi_\mu &= D_\mu \epsilon - \frac{1}{24} e^\sigma \gamma^{\rho\sigma\tau} \gamma_\mu G_{\rho\sigma\tau} \epsilon , \\ \delta \chi &= -\frac{1}{2} \gamma^\mu \partial_\mu \sigma \epsilon - \frac{1}{12} e^\sigma \gamma^{\rho\sigma\tau} G_{\rho\sigma\tau} \epsilon , \\ \delta B_{\mu\nu} &= i e^{-\sigma} \left(\bar{\epsilon} \gamma_{[\mu} \psi_{\nu]} + \frac{1}{2} \bar{\epsilon} \gamma_{\mu\nu} \chi \right) - A_{[\mu}^{r'} \delta A_{\nu]}^{r'} , \\ \delta \sigma &= -i \bar{\epsilon} \chi , \\ \delta A_\mu^{r'} &= i e^{-\sigma/2} \bar{\epsilon} \gamma_\mu \lambda^{r'} , \\ \delta \lambda^{r'} &= -\frac{1}{4} e^{\sigma/2} \gamma^{\mu\nu} F_{\mu\nu}^{r'} \epsilon + \frac{1}{2} e^{-\sigma/2} \hat{C}^{\hat{i}\hat{j}r'} \Gamma_{\hat{i}\hat{j}} \epsilon , \\ \hat{L}_{\hat{i}}^{\hat{I}} \delta \hat{L}_{\hat{I}}^{\hat{r}} &= -\bar{\epsilon} \Gamma_{\hat{i}} \psi^{\hat{r}} , \\ \delta \psi^{\hat{r}} &= \frac{i}{2} \gamma^\mu \hat{P}_\mu^{\hat{r}\hat{s}} \bar{\Gamma}_{\hat{i}} \epsilon . \end{aligned} \quad (5.10)$$

The relation between the supersymmetric variation of the enlarged coset representative and those involving the $SO(n, 3)/SO(n) \times SO(3)$ coset representative, the dilaton and axions

is similar to the relations in (5.1) for the corresponding Maurer-Cartan forms, and field strengths, since $L^{-1}dL$ has the same decomposition as $L^{-1}\delta L$. Finally, we note that the above results for the matter coupled gauged $N = (1, 0)$ supergravity in $6D$ are in accordance with the results given in [1].

5.2 Hidden Symmetry in the Iwasawa Gauge

5.2.1 The Ungauged Sector

In order to comply with the standard notation for the Iwasawa decomposition of $SO(p, q)$, we switch from our coset representative to its transpose as $L = \mathcal{V}^T$. Following [20, 21], we then parametrize the coset $SO(p+3, 3)/SO(p+3) \times SO(3)$ as

$$\mathcal{V} = e^{\frac{1}{2}\vec{\varphi} \cdot \vec{H}} e^{C^i{}_j E_i{}^j} e^{\frac{1}{2}A_{ij} V^{ij}} e^{B^{\underline{r}\underline{s}} U_{\underline{r}\underline{s}}}, \quad i = 1, \dots, 3, \quad \underline{r} = 1, \dots, p, \quad (5.11)$$

where $(U_{\underline{r}\underline{s}}, V^{ij}, E_i{}^j, \vec{H})$ with $i < j$ and $V^{ij} = -V^{ji}$, are the generators of the $3(p+3)$ dimensional solvable subalgebra of $SO(p+3, 3)$ multiplying the corresponding scalar fields and $\vec{\varphi} \cdot \vec{H}$ stands for $\varphi^i H_i$.

Using the commutation rules of the generators given in Appendix D, one finds [20]

$$\mathcal{V} = \left(\begin{array}{c|c|c} e^{\frac{1}{2}\vec{c}_i \cdot \vec{\varphi}} \gamma^j{}_i & e^{\frac{1}{2}\vec{c}_i \cdot \vec{\varphi}} \gamma^k{}_i B^s{}_k & e^{\frac{1}{2}\vec{c}_i \cdot \vec{\varphi}} \gamma^k{}_i \left(A_{kj} + \frac{1}{2} B^a{}_k B^a{}_j \right) \\ \hline 0 & \delta_{\underline{r}\underline{s}} & B^{\underline{r}}{}_j \\ \hline 0 & 0 & e^{-\frac{1}{2}\vec{c}_i \cdot \vec{\varphi}} \tilde{\gamma}^i{}_j \end{array} \right), \quad (5.12)$$

where \vec{c}_i is defined in (D.5), and

$$\tilde{\gamma}^i{}_j = \delta^i{}_j + C^i{}_j, \quad \gamma^i{}_k \tilde{\gamma}^k{}_j = \delta^i{}_j. \quad (5.13)$$

The inverse of \mathcal{V} can be computed from the defining relation $\mathcal{V}^T \Omega \mathcal{V} = \Omega$ and is given by:

$$\mathcal{V}^{-1} = \left(\begin{array}{c|c|c} e^{-\frac{1}{2}\vec{c}_j \cdot \vec{\varphi}} \tilde{\gamma}^j{}_i & -B^{\underline{s}}{}_i & e^{\frac{1}{2}\vec{c}_j \cdot \vec{\varphi}} \gamma^k{}_j \left(A_{ki} + \frac{1}{2} B^a{}_k B^a{}_i \right) \\ \hline 0 & \delta_{\underline{r}\underline{s}} & -e^{\frac{1}{2}\vec{c}_j \cdot \vec{\varphi}} \gamma^k{}_j B^{\underline{r}}{}_k \\ \hline 0 & 0 & e^{\frac{1}{2}\vec{c}_j \cdot \vec{\varphi}} \gamma^i{}_j \end{array} \right). \quad (5.14)$$

In equations (5.12) and (5.14) the indices (i, \underline{r}) label the rows and (j, \underline{s}) label the column. The Iwasawa gauge means setting the scalars corresponding to the maximal compact subalgebra equal to zero. Under the action of the global G transformations from the right, the coset representative will not remain in the Iwasawa gauge but can be brought back

to that form by a compensating h transformation from the left, namely, $\mathcal{V}g = h\mathcal{V}'$. The Maurer-Cartan form $d\mathcal{V}\mathcal{V}^{-1}$ can be decomposed into two parts one of which transforms homogeneously under h and the other one transforms as an h -valued gauge field:

$$\begin{aligned} P &= d\mathcal{V}\mathcal{V}^{-1} + \left(d\mathcal{V}\mathcal{V}^{-1}\right)^T : & P &\rightarrow hPh^{-1} , \\ Q &= d\mathcal{V}\mathcal{V}^{-1} - \left(d\mathcal{V}\mathcal{V}^{-1}\right)^T : & Q &\rightarrow dh h^{-1} + hQh^{-1} . \end{aligned} \quad (5.15)$$

Both of these are, of course, manifestly invariant under the global g transformations. The key building block in writing down the action in Iwasawa gauge is the Maurer-Cartan form [20]

$$\partial_\mu \mathcal{V}\mathcal{V}^{-1} = \frac{1}{2} \partial_\mu \vec{\varphi} \cdot \vec{H} + \sum_{i < j} \left(e^{\frac{1}{2} \vec{a}_{ij} \cdot \vec{\varphi}} F_{\mu ij} V^{ij} + e^{\frac{1}{2} \vec{b}_{ij} \cdot \vec{\varphi}} \mathcal{F}_{\mu j}^i E_i^j \right) + \sum_{i, \underline{x}} F_{\mu i \underline{x}} U^{i \underline{x}} , \quad (5.16)$$

where \vec{a}_{ij} and \vec{b}_{ij} are defined in (D.5) and

$$\begin{aligned} F_{\mu ij} &= \gamma^k_i \gamma^\ell_j (\partial_\mu A_{k\ell} - B^{\underline{a}}_k \partial_\mu B^{\underline{a}}_\ell) , \\ F_{\mu i \underline{x}} &= \gamma^j_i \partial_\mu B_{j \underline{x}} , \\ \mathcal{F}_{\mu j}^i &= \gamma^k_j \partial_\mu C^i_k , \end{aligned} \quad (5.17)$$

and it is understood that $i < j$. Other building blocks for the action are the field strengths for the axions defined as

$$\mathcal{V} \partial_\mu \Phi = \begin{pmatrix} F_{\mu i} \\ F_\mu^{\underline{x}} \\ \mathcal{F}_\mu^i \end{pmatrix} , \quad \Phi = \begin{pmatrix} A_i \\ B^{\underline{x}} \\ C^i \end{pmatrix} , \quad (5.18)$$

where A_i and $B^{\underline{x}}$ form a $(p+3)$ dimensional representation of $SO(p+3) \subset SO(p+3, 3)$, and

$$\begin{aligned} F_{\mu i} &= \gamma^j_i \left(\partial_\mu A_j + B^{\underline{x}}_j \partial_\mu B^{\underline{x}} + A_{jk} \partial_\mu C^k + \frac{1}{2} B^{\underline{x}}_j B^{\underline{x}}_k \partial_\mu C^k \right) , \\ F_\mu^{\underline{x}} &= \partial_\mu B^{\underline{x}} + B^{\underline{x}}_i \partial_\mu C^i , \\ \mathcal{F}_\mu^i &= \tilde{\gamma}^i_j \partial_\mu C^j . \end{aligned} \quad (5.19)$$

With these definitions, the bosonic part of our 6D Lagrangian that contains the $(4p+4)$ scalar fields, which we shall call $\mathcal{L}_{G/H}$, takes the form

$$\begin{aligned} e^{-1} \mathcal{L}_{G/H} &= -\frac{1}{4} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{8} \text{tr} \left(d\mathcal{V}\mathcal{V}^{-1} \right) \left(d\mathcal{V}\mathcal{V}^{-1} + (d\mathcal{V}\mathcal{V}^{-1})^T \right) \\ &\quad - \frac{1}{4} e^{2\varphi} (\mathcal{V} \partial_\mu \Phi)^T (\mathcal{V} \partial^\mu \Phi) . \end{aligned} \quad (5.20)$$

More explicitly, this can be written as [20]

$$\begin{aligned}
e^{-1}\mathcal{L}_{G/H} = & -\frac{1}{4}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{8}\sum_{i<j}\left(e^{\vec{a}_{ij}\cdot\vec{\varphi}}F_{\mu ij}F^{\mu ij} + e^{\vec{b}_{ij}\cdot\vec{\varphi}}\mathcal{F}_\mu{}^i{}_j\mathcal{F}^\mu{}^j{}_i\right) - \sum_{i,r}\frac{1}{8}e^{\vec{c}_i\cdot\vec{\varphi}}F_{\mu i\mathbf{r}}F^{\mu i\mathbf{r}} \\
& -\frac{1}{4}e^{2\varphi}\left(e^{\vec{c}_i\cdot\vec{\varphi}}F_{\mu i}F^{\mu i} + e^{-\vec{c}_i\cdot\vec{\varphi}}\mathcal{F}_\mu{}^i\mathcal{F}_i{}^\mu + F_\mu^rF_r^\mu\right) .
\end{aligned} \tag{5.21}$$

The idea is now to combine the dilaton and axionic scalar field strengths (5.19) with the scalar field strengths for $SO(p+3)/SO(p)\times SO(3)$ defined in (5.17) to express them all as the scalar field strengths of the enlarged coset $SO(p+4,4)/SO(p+4)\times SO(4)$. As is well known, this is indeed possible and to this end we need to make the identifications

$$\begin{aligned}
F_{\mu i} &= \frac{1}{\sqrt{2}}F_{\mu i4} , \\
F_\mu^r &= \frac{1}{\sqrt{2}}F_{\mu 4}^r , \\
\mathcal{F}_\mu^i &= \frac{1}{\sqrt{2}}\mathcal{F}_{\mu 4}^i .
\end{aligned} \tag{5.22}$$

The quantities on the right hand side are restrictions of the Maurer-Cartan form based on the enlarged coset $SO(p+4,4)/SO(p+4)\times SO(3)$ defined as

$$\partial_\mu\widehat{\mathcal{V}}\widehat{\mathcal{V}}^{-1} = \frac{1}{2}\partial_\mu\varphi^\alpha H_\alpha + \sum_{\alpha<\beta}\left(e^{\frac{1}{2}\vec{a}_{\alpha\beta}\cdot\vec{\varphi}}F_{\mu\alpha\beta}V^{\alpha\beta} + e^{\frac{1}{2}\vec{b}_{\alpha\beta}\cdot\vec{\varphi}}\mathcal{F}_\mu{}^\alpha{}_\beta E_\alpha{}^\beta\right) + \sum_{\alpha,r}e^{\frac{1}{2}\vec{c}_\alpha\cdot\vec{\varphi}}F_{\mu\alpha\mathbf{r}}U^{\alpha\mathbf{r}} , \tag{5.23}$$

where $\widehat{\mathcal{V}}$ is defined as in (5.12) and $(F_{\mu\alpha\beta}, \mathcal{F}_\mu{}^\alpha{}_\beta, F_{\mu\alpha\mathbf{r}})$ as in (5.17), and $\vec{a}_{\alpha\beta}, \vec{b}_{\alpha\beta}, \vec{c}_\alpha$ as in (D.5), with the *3-valued indices replaced by the 4-valued indices everywhere*. Equations (5.22) have a solution given by

$$\begin{aligned}
A_i &= \frac{1}{\sqrt{2}}\left(A_{i4} - A_{ij}\gamma^j{}_k C^k{}_4 - \frac{1}{2}B_i^r B_r{}_4 + \frac{1}{2}B_i^r B_r{}^j{}_j C^k{}_4\right) , \\
B_i^r &= \frac{1}{\sqrt{2}}\left(B_r{}_4 - B_i^r \gamma^i{}_j C^j{}_4\right) , \\
C^i &= \frac{1}{\sqrt{2}}\gamma^i{}_j C^j{}_4 , \\
\varphi &= \frac{1}{\sqrt{2}}\varphi_4 .
\end{aligned} \tag{5.24}$$

The identifications (5.22) (where $r \rightarrow \mathbf{r}, i$), together with (4.9), (5.18), (5.23), (5.16), (4.7) (with $A_\mu = 0$), (5.15) and (D.5) (defined for 3-valued and 4-valued indices similarly), provide the proof of (5.1) used to show the hidden symmetry. Using (5.22), the Lagrangian $\mathcal{L}_{G/H}$

can be written as the $SO(p+4,4)/SO(p+4) \times SO(4)$ sigma model:

$$\begin{aligned}
e^{-1} \mathcal{L}_{G/H} &= -\frac{1}{8} \text{tr} \left(d\hat{\mathcal{V}} \hat{\mathcal{V}}^{-1} \right) \left(d\hat{\mathcal{V}} \hat{\mathcal{V}}^{-1} + (d\hat{\mathcal{V}} \hat{\mathcal{V}}^{-1})^T \right) \\
&= -\frac{1}{8} \partial \varphi^\alpha \partial^\mu \varphi_\alpha - \frac{1}{8} \sum_{\alpha < \beta} \left(e^{\vec{a}_{\alpha\beta} \cdot \vec{\varphi}} F_\mu^{\alpha\beta} F_{\alpha\beta}^\mu + e^{\vec{b}_{\alpha\beta} \cdot \vec{\varphi}} \mathcal{F}_{\mu\alpha}^\beta \mathcal{F}^{\mu\alpha}_\beta \right) - \frac{1}{8} \sum_{\alpha, r} e^{\vec{c}_\alpha \cdot \vec{\varphi}} F_{\mu\alpha r} F^{\mu\alpha r},
\end{aligned} \tag{5.25}$$

where $a_{\alpha\beta}, b_{\alpha\beta}, c_\alpha$ are defined as in (D.5) with the indices $i, j = 1, 2, 3$ replaced by $\alpha, \beta = 1, \dots, 4$.

5.2.2 Gauging and the C-functions

In order to justify the identifications (5.2) of the S -functions as certain components of the C -functions associated with the enlarged coset space, we need to study these functions in the Iwasawa gauge. Of the four supergravity models in $6D$ that have nonvanishing gauge functions, two of them, namely models (II) and (IV), see section 3.2, are coupled to one special linear multiplet and as such they deserve separate treatment. In both of these cases, the gauge functions obey the relations (4.11). We begin by showing how these relations follow from the C -functions associated with the $U(1)_R$ or $Sp(1)_R$ gauged $SO(4,1)/SO(4)$ sigma model.

The coset representative for $SO(4,1)/SO(4)$ in the Iwasawa gauge takes the form

$$\mathcal{V} = \left(\begin{array}{c|c|c} e^\varphi & e^\varphi \Phi^i & \frac{1}{2} e^\varphi \Phi^2 \\ \hline 0 & \delta_{ij} & \Phi^i \\ \hline 0 & 0 & e^{-\varphi} \end{array} \right), \tag{5.26}$$

where $\Phi^2 = \Phi^i \Phi_i$. Note that we have made the identification $B_{1r} \rightarrow \Phi_i$ already. Given that the $Sp(1)_R$ or $U(1)_R$ generators $T^{r'}$ are of the form

$$T^{r'} = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & T^{r'} & 0 \\ \hline 0 & 0 & 0 \end{array} \right), \tag{5.27}$$

we find that the C function based on the enlarged coset is given by

$$C^{r'} = \mathcal{V} T^{r'} \mathcal{V}^{-1} = \left(\begin{array}{c|c|c} 0 & -T_{ij}^{r'} \Phi^j & 0 \\ \hline 0 & T_{ij}^{r'} & -T_{ij}^{r'} \Phi^j \\ \hline 0 & 0 & 0 \end{array} \right). \tag{5.28}$$

Comparing with the relations given in (4.11), and recalling that $T_{ij}^{r'} \sim \epsilon_{r'ij}$, we see that indeed the projection of the C function based on the enlarged coset as defined above does

produce the C and S functions obtained from the chiral reduction, as was assumed in the previous section in (5.2). In the notation of Appendix D, the C function is obtained from projection by T_{ij} , and the S -function from projection by U_i .

Next, we consider the remaining two supergravities with nontrivial gauge function, namely models (V) and (VI), see section 3.2, with hidden symmetry chains shown in (3.34) and (3.36). Prior to uncovering the hidden symmetry, the C and S functions occurring in the Lagrangian (4.3) and in the supersymmetry transformations (4.6) are $C^{ijr'}$, $C^{irr'}$, $S^{ir'}$ and $S^{rr'}$. Using (4.10), and the fact that $f_{r'I}{}^J \sim (T_{r'})_I{}^J$, we deduce the definitions

$$\begin{aligned}\vec{C}^{r'}(H) &= \text{tr} \left(\mathcal{V} T^{r'} \mathcal{V}^{-1} \right) \vec{H} , \\ C_i^{jr'}(E) &= \text{tr} \left(\mathcal{V} T^{r'} \mathcal{V}^{-1} \right) E_i^j , \\ C_{ij}{}^{r'}(V) &= \text{tr} \left(\mathcal{V} T^{r'} \mathcal{V}^{-1} \right) V^{ij} , \\ C_{i\underline{r}}{}^{r'}(U) &= \text{tr} \left(\mathcal{V} T^{r'} \mathcal{V}^{-1} \right) U_{i\underline{r}} ,\end{aligned}\tag{5.29}$$

for the C functions in the coset direction,

$$C^{ijr'}(X) = \text{tr} \left(\mathcal{V} T^{r'} \mathcal{V}^{-1} \right) X^{ij} ,\tag{5.30}$$

for the C function in the $SO(3)$ direction, and

$$S^{r'} = \begin{pmatrix} S_i^{r'} \\ S^{rr'} \\ S^{ir'} \end{pmatrix} = \sqrt{2} e^\varphi \mathcal{V} T^{r'} \Phi ,\tag{5.31}$$

for the S functions. For $SO(p, q)/SO(p) \times SO(q)$ with $p \geq q$, we have $i = 1, \dots, q$ and $\underline{r} = 1, \dots, p - q$. To show that the C and S functions defined above combine to give the \hat{C} functions for the enlarged coset, we begin with the observation that the gauge group lies in the H^i and $T_{\underline{r}s}$ directions. Thus we can denote the full gauge symmetry generator that acts on the enlarged coset representative \mathcal{V} as

$$T^{r'} = \left(\begin{array}{c|c|c} H^{r'} & 0 & 0 \\ \hline 0 & T^{r'} & 0 \\ \hline 0 & 0 & H^{r'} \end{array} \right) ,\tag{5.32}$$

where $H_{\alpha\beta}^{r'}$ ($\alpha, \beta = 1, \dots, q + 1$) is symmetric and $T_{rs}^{r'}$ ($r, s = 1, \dots, p - q$) is antisymmetric. Defining the C functions for the coset $SO(p + 1, q + 1)/SO(p + 1) \times SO(q + 1)$ as in (5.29)

with the index $i = 1, \dots, q$ replaced by $\alpha = 1, \dots, q + 1$ and $T^{r'}$ defined in (5.32), we find

$$\begin{aligned}
\vec{C}^{r'}(H) &= \gamma^\gamma_\alpha H_\gamma^{r'\delta} \tilde{\gamma}^\alpha_\delta \vec{c}_\alpha, \\
C^\beta_{\alpha}{}^{r'}(E) &= e^{\frac{1}{2}(\vec{c}_\alpha - \vec{c}_\beta) \cdot \vec{\varphi}} \gamma^\gamma_\alpha H_\gamma^{r'\delta} \tilde{\gamma}^\beta_\delta, \\
\mathcal{C}_{\alpha\beta r'}(V) &= e^{\frac{1}{2}(\vec{c}_\alpha - \vec{c}_\beta) \cdot \vec{\varphi}} \gamma^\gamma_{[\alpha} \gamma^\delta_{\beta]} \left(H_\gamma^{r'\eta} (A_{\delta\eta} + \frac{1}{2} B_\delta^r B_\eta^r) - T_{rs}^{r'} B_\gamma^r B_\delta^s \right), \\
C_{\alpha r}{}^{r'}(U) &= -e^{\frac{1}{2}\vec{c}_\alpha \cdot \vec{\varphi}} \gamma^\beta_\alpha \left(H_\beta^{r'\gamma} B_{\gamma r} + T_{rs}^{r'} B_\beta^s \right). \tag{5.33}
\end{aligned}$$

Using (5.24), the above quantities reduce to those for the $SO(p+3, 3)/SO(p+3) \times SO(3)$ coset upon restriction of the 4-valued α, β indices to 3-valued (i, j) indices, $C^{4r'}(H) = 0$ and

$$\begin{aligned}
C_{4i}^{r'}(E) &= \sqrt{2} e^\varphi S^{r'}_i, \\
C_{4i}^{r'}(V) &= \sqrt{2} e^\varphi S^{r'}_i, \\
C_{4r}^{r'}(U) &= -\sqrt{2} e^\varphi S^{r'}_r. \tag{5.34}
\end{aligned}$$

These identifications, upon comparing the definitions (5.29), (5.30) and (5.31) with (4.10), provide the proof of the relations (5.2) used in showing the hidden symmetry. In doing so, note that $C^{irr'} \rightarrow (C^{irr'}, C^{i,jr'})$ and $S^{rr'} \rightarrow (S^{rr'}, S^{ir'})$ and that $C^{i,jr'}$ has components in the (\vec{H}, E, V, U, X) directions. Note also that, having proven the relations (5.1) and (5.2), it follows that not only the bosonic part of the $6D$ Lagrangian, (5.3), but also its part that contains the fermions, namely, (5.5), exhibits correctly the enlarged coset structure. This concludes the demonstration of the enlarged coset structure in the $6D$ models with nontrivial gauge functions.

6 Comments

We have reduced the half-maximal $7D$ supergravity with specific noncompact gaugings coupled to a suitable number of vector multiplets on a circle to $6D$ and chirally truncated it to $N = (1, 0)$ supergravity such that a R -symmetry gauging survives. These are referred to as the $SO(3, 1)$, $SO(3, 1)$ and $SO(2, 2)$ models, and their field content and gauge symmetries are summarized in the Introduction. These models, in particular, feature couplings to p linear multiplets whose scalar fields parametrize the coset $SO(p, 3)/SO(p) \times SO(3)$, a dilaton and $(p+3)$ axions, for $p \leq 1$. The value of p is restricted in the case of chiral circle reductions that maintain R -symmetry gauging, but it is arbitrary otherwise. We have exhibited in

the full model, including the fermionic contributions, how these fields can be combined to parametrize an enlarged coset $SO(p+1,4)/SO(p+1) \times SO(4)$ whose abelian isometries correspond to the $(p+3)$ axions. In the ungauged $6D$ models obtained, we have dualized the axions to 4-form potentials, thereby obtaining the coupling of p linear multiplets and one special linear multiplet to chiral $6D$ supergravity (see Appendix C).

In this paper, we have also shown that, contrary to the claims made in the literature [18], the 2-form potential in the gauged $7D$ supergravity can be dualized to a 3-form potential even in the presence of couplings to an arbitrary number of vector multiplets.

Our results for the R -symmetry gauged reduction of certain noncompact gauged $7D$ supergravities are likely to play an important role in finding the string/M-theory origin of the *gauged and anomaly-free* $N = (1,0)$ supergravities in $6D$ which has been a notoriously challenging problem so far. This is due to the fact that at least two of the $7D$ models we have encountered, namely the $SO(3,1)$ and $SO(2,2)$ gauged $7D$ models, are known to have a string/M-theory origin. Therefore, what remains to be understood is the introduction of the matter couplings in $6D$ that are needed for anomaly freedom. A natural approach for achieving this is to associate our chiral reduction with boundary conditions to be imposed on the fields of the $7D$ model formulated on a manifold with boundary [22].

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A The Dual Gauged 7D Model with Matter Couplings and Topological Mass Term

The 2-form potential occurring in the model of [13] given above can easily be dualized [22] to a 3-form potential. To do this, one adds the total derivative term to obtain the new Lagrangian

$$\mathcal{L}_3 = \mathcal{L} - \frac{1}{144} \epsilon^{\mu_1 \dots \mu_7} H_{\mu_1 \dots \mu_4} \left(G_{\mu_5 \dots \mu_7} + \frac{3}{\sqrt{2}} \omega_{\mu_5 \dots \mu_7}^0 \right), \quad (\text{A.1})$$

where

$$H_{\mu\nu\rho\sigma} = 4\partial_{[\mu} C_{\nu\rho\sigma]}. \quad (\text{A.2})$$

We can treat G as an independent field because the C -field equation will impose the correct Bianchi identity that implies the correct form of G given in the previous section. Thus, treating G as an independent field, its field equation gives

$$G_{\mu\nu\rho} = -\frac{1}{24} e^{-2\sigma} e \epsilon_{\mu\nu\rho\sigma_1 \dots \sigma_4} H^{\sigma_1 \dots \sigma_4} + \frac{i}{4\sqrt{2}} e^{-\sigma} X_{\mu\nu\rho}. \quad (\text{A.3})$$

Using this result in the Lagrangian given in (2.8), one finds

$$\begin{aligned} \mathcal{L}_3 &= \mathcal{L}' - \frac{1}{48} e e^{-2\sigma} H_{\mu\nu\rho\sigma} H^{\mu\nu\rho\sigma} - \frac{1}{48\sqrt{2}} \epsilon^{\mu_1 \dots \mu_7} H_{\mu_1 \dots \mu_4} \omega_{\mu_5 \dots \mu_7}^0 \\ &\quad - \frac{i}{576\sqrt{2}} e^{-\sigma} \epsilon^{\mu_1 \dots \mu_7} H_{\mu_1 \dots \mu_4} X_{\mu_5 \dots \mu_7}, \end{aligned} \quad (\text{A.4})$$

where \mathcal{L}' is the G -independent part of (2.8). For the readers convenience, we explicitly give the dual Lagrangian $\mathcal{L}_3 = \mathcal{L}_{3B} + \mathcal{L}_{3F}$ where

$$\begin{aligned} e^{-1} \mathcal{L}_{3B} &= \frac{1}{2} R - \frac{1}{4} e^\sigma a_{IJ} F_{\mu\nu}^I F^{\mu\nu J} - \frac{1}{48} e e^{-2\sigma} H_{\mu\nu\rho\sigma} H^{\mu\nu\rho\sigma} - \frac{1}{48\sqrt{2}} \epsilon^{\mu_1 \dots \mu_7} H_{\mu_1 \dots \mu_4} \omega_{\mu_5 \dots \mu_7}^0 \\ &\quad - \frac{5}{8} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} P_\mu^{ir} P_{ir}^\mu - \frac{1}{4} e^{-\sigma} \left(C^{ir} C_{ir} - \frac{1}{9} C^2 \right), \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} e^{-1} \mathcal{L}_{3F} &= -\frac{i}{2} \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{5i}{2} \bar{\chi} \gamma^\mu D_\mu \chi - \frac{i}{2} \bar{\lambda}^r \gamma^\mu D_\mu \lambda_r - \frac{5i}{4} \bar{\chi} \gamma^\mu \gamma^\nu \psi_\mu \partial_\nu \sigma - \frac{1}{2} \bar{\lambda}^r \sigma^i \gamma^\mu \gamma^\nu \psi_\mu P_{\nu ri} \\ &\quad + \frac{i}{96\sqrt{2}} e^\sigma H_{\mu\nu\rho\sigma} X^{\mu\nu\rho\sigma} + \frac{1}{8} e^{\sigma/2} F_{\mu\nu}^i X_i^{\mu\nu} - \frac{i}{4} e^{\sigma/2} F_{\mu\nu}^r X_r^{\mu\nu} \\ &\quad - \frac{i\sqrt{2}}{24} e^{-\sigma/2} C \left(\bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu + 2\bar{\psi}_\mu \gamma^\mu \chi + 3\bar{\chi} \chi - \bar{\lambda}^r \lambda_r \right) \\ &\quad + \frac{1}{2\sqrt{2}} e^{-\sigma/2} C_{ir} \left(\bar{\psi}_\mu \sigma^i \gamma^\mu \lambda^r - 2\bar{\chi} \sigma^i \lambda^r \right) + \frac{1}{2} e^{-\sigma/2} C_{rsi} \bar{\lambda}^r \sigma^i \lambda^s, \end{aligned} \quad (\text{A.6})$$

and where the fermionic bilinears are defined as

$$\begin{aligned} X^{\mu\nu\rho\sigma} &= \bar{\psi}^\lambda \gamma_{[\lambda} \gamma^{\mu\nu\rho\sigma} \gamma_{\tau]} \psi^\tau + 4\bar{\psi} \lambda \gamma^{\mu\nu\rho\sigma} \gamma^\lambda \chi - 3\bar{\chi} \gamma^{\mu\nu\rho\sigma} \chi + \bar{\lambda}^a \gamma^{\mu\nu\rho\sigma} \lambda_a, \\ X^{i\mu\nu} &= \bar{\psi}^\lambda \sigma^i \gamma_{[\lambda} \gamma^{\mu\nu} \gamma_{\tau]} \psi^\tau - 2\bar{\psi} \lambda \sigma^i \gamma^{\mu\nu} \gamma^\lambda \chi + 3\bar{\chi} \sigma^i \gamma^{\mu\nu} \chi - \bar{\lambda}^r \sigma^i \gamma^{\mu\nu} \lambda_r, \\ X^{r\mu\nu} &= \bar{\psi} \lambda \gamma^{\mu\nu} \gamma^r \lambda^r + 2\bar{\chi} \gamma^{\mu\nu} \lambda^r. \end{aligned} \quad (\text{A.7})$$

The supersymmetry transformation rules are

$$\begin{aligned}
\delta e_\mu^m &= i\bar{\epsilon}\gamma^m\psi_\mu, \\
\delta\psi_\mu &= 2D_\mu\epsilon - \frac{\sqrt{2}}{30}e^{-\sigma/2}C\gamma_\mu\epsilon \\
&\quad - \frac{1}{240\sqrt{2}}e^{-\sigma}H_{\rho\sigma\lambda\tau}\left(\gamma_\mu\gamma^{\rho\sigma\lambda\tau} + 5\gamma^{\rho\sigma\lambda\tau}\gamma_\mu\right)\epsilon - \frac{i}{20}e^{\sigma/2}F_{\rho\sigma}^i\sigma^i(3\gamma_\mu\gamma^{\rho\sigma} - 5\gamma^{\rho\sigma}\gamma_\mu)\epsilon, \\
\delta\chi &= -\frac{1}{2}\gamma^\mu\partial_\mu\sigma\epsilon - \frac{i}{10}e^{\sigma/2}F_{\mu\nu}^i\sigma^i\gamma^{\mu\nu}\epsilon - \frac{1}{60\sqrt{2}}e^{-\sigma}H_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}\epsilon + \frac{\sqrt{2}}{30}e^{\sigma/2}C\epsilon, \\
\delta C_{\mu\nu\rho} &= e^\sigma\left(\frac{3i}{\sqrt{2}}\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]} - i\sqrt{2}\bar{\epsilon}\gamma_{\mu\nu\rho}\chi\right), \tag{A.8}
\end{aligned}$$

$$\begin{aligned}
\delta\sigma &= -2i\bar{\epsilon}\chi, \\
\delta A_\mu^I &= -e^{-\sigma/2}\left(\bar{\epsilon}\sigma^i\psi_\mu + \bar{\epsilon}\sigma^i\gamma_\mu\chi\right)L_i^I + ie^{-\sigma/2}\bar{\epsilon}\gamma_\mu\lambda^r L_r^I, \\
\delta L_I^r &= \bar{\epsilon}\sigma^i\lambda^r L_I^i, \quad \delta L_I^i = \bar{\epsilon}\sigma^i\lambda_r L_I^r, \tag{A.9} \\
\delta\lambda^r &= -\frac{1}{2}e^{\sigma/2}F_{\mu\nu}^r\gamma^{\mu\nu}\epsilon + i\gamma^\mu P_\mu^{ir}\sigma^i\epsilon - \frac{i}{\sqrt{2}}e^{-\sigma/2}C^{ir}\sigma^i\epsilon.
\end{aligned}$$

The supersymmetry transformation rule for the 3-form potential can be obtained from the supersymmetry of the G -field equation (A.3). Indeed, it is sufficient to check the cancellation of the $\partial_\mu\epsilon$ terms to determine the supersymmetry variation of the 3-form potential. If we set to zero all the vector multiplet fields, the above Lagrangian and transformation rules become those of $SU(2)$ gauged pure half-maximal supergravity [23], which in turn admits a topological mass term for the 3-form potential in a supersymmetric fashion that involves a new constant parameter [23]. In [18], it has been argued that the gauged theory in presence of the coupling to vector multiplets does not admit a topological mass term. However, we have found that this is not the case. Indeed, we have found that one can add the following Lagrangian to \mathcal{L}_3 given in (A.4):

$$\begin{aligned}
e^{-1}\mathcal{L}_h &= \frac{he^{-1}}{36}\epsilon^{\mu_1\cdots\mu_7}H_{\mu_1\cdots\mu_4}C_{\mu_5\cdots\mu_7} + \frac{4\sqrt{2}}{3}he^{3\sigma/2}C - 16ih^2e^{4\sigma} \\
&\quad ihe^{2\sigma}\left(-\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu + 8\bar{\psi}_\mu\gamma^\mu\chi + 27\bar{\chi}\chi - \bar{\lambda}^r\lambda_r\right). \tag{A.10}
\end{aligned}$$

Note that the coupling of matter to the model with topological mass term has led to the dressing up of the term $he^{3\sigma/2}$ present in that model by C as shown in the second term on the right hand side of (A.10). The second ingredient to make the supersymmetry work is the term $he^{2\sigma}\bar{\lambda}^r\lambda_r$ in (A.10)⁹. The action for the total Lagrangian

$$\mathcal{L}_{new} = \mathcal{L}_3 + \mathcal{L}_h \tag{A.11}$$

⁹The obstacle reported in [18] in coupling matter in presence of the topological terms may be due to the fact that these ingredients were not considered.

is invariant under the supersymmetry transformation rules described above with the following new h -dependent terms:

$$\begin{aligned}\delta_h \psi_\mu &= -\frac{4}{5} h e^{2\sigma} \gamma_\mu \epsilon , \\ \delta_h \chi &= -\frac{16}{5} h e^{2\sigma} \epsilon .\end{aligned}\tag{A.12}$$

For comparison with [23], we extract the potential and all the mass terms, and write it as

$$\begin{aligned}\Delta \mathcal{L} &= 60m^2 - 10 \left(m + 2he^{2\sigma} \right)^2 + \frac{5im}{2} \bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu - 5i \left(m + 2he^{2\sigma} \right) \bar{\psi} \gamma^\mu \chi \\ &\quad + 5i \left(\frac{3}{2}m + 6he^{2\sigma} \right) \bar{\chi} \chi - \frac{i}{2} \left(5m + 4he^{2\sigma} \right) \bar{\lambda}^r \lambda_r - \frac{1}{4} e^{-\sigma} C^{ir} C_{ir} \\ &\quad + \frac{1}{2\sqrt{2}} e^{-\sigma/2} C_{ir} \left(\bar{\psi}_\mu \sigma^i \gamma^\mu \lambda^r - 2\bar{\chi} \sigma^i \lambda^r \right) + \frac{1}{2} e^{-\sigma/2} C_{rsi} \bar{\lambda}^r \sigma^i \lambda^s ,\end{aligned}\tag{A.13}$$

where we have defined

$$m = -\frac{1}{30\sqrt{2}} C e^{-\sigma/2} - \frac{2}{5} h e^{2\sigma} ,\tag{A.14}$$

so that

$$\begin{aligned}\delta' \psi_\mu &= 2m \gamma_\mu \epsilon , \\ \delta' \chi &= -2(m + 2he^{2\sigma}) \epsilon .\end{aligned}\tag{A.15}$$

In the absence of matter couplings, the above result has exactly the same structure as that of [23] but the coefficients differ, even after taking into account the appropriate constant rescalings of fields and parameters due to convention differences.

B The Map Between $SL(4, R)/SO(4)$ and $SO(3, 3)/SO(3) \times SO(3)$

Let us denote the $SL(4, R)/SO(4)$ coset representative by \mathcal{V}_α^R which is a 4×4 unimodular real matrix with inverse \mathcal{V}_R^α :

$$\mathcal{V}_R^\alpha \mathcal{V}_\alpha^S = \delta_R^S , \quad \alpha = 1, \dots, 4, \quad R = 1, \dots, 4 .\tag{B.1}$$

The map between \mathcal{V}_α^R and the $SO(3, 3)/SO(3) \times SO(3)$ coset representative L_I^A can be written as

$$L_I^A = \frac{1}{4} \Gamma_I^{\alpha\beta} \eta_{RS}^A \mathcal{V}_\alpha^R \mathcal{V}_\beta^S \equiv \frac{1}{4} \mathcal{V} \Gamma_I \eta^A \mathcal{V} ,\tag{B.2}$$

where Γ^I and η^A are the chirally projected $SO(3, 3)$ Dirac matrices which satisfy [24]

$$(\Gamma^I)_{\alpha\beta} (\Gamma^J)^{\alpha\beta} = -4\eta^{IJ} , \quad (\Gamma^I)_{\alpha\beta} (\Gamma_I)_{\gamma\delta} = -2\epsilon_{\alpha\beta\gamma\delta} ,\tag{B.3}$$

where η_{IJ} as well as η_{AB} have signature $(---++)$. Similar identities are satisfied by $(\eta^A)_{RS}$. Both Γ^I and η^A are antisymmetric. Pairs of antisymmetric indices are raised and lowered by the ϵ tensor:

$$V^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}V_{\gamma\delta} , \quad V_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}V^{\gamma\delta} . \quad (\text{B.4})$$

Since \mathcal{V} is real, the Γ and η -matrices must be real as well. A convenient such representation is given by

$$\Gamma^I \equiv (\Gamma^I)_{\alpha\beta} = \begin{pmatrix} \alpha^i \\ \beta^r \end{pmatrix} , \quad (\Gamma^I)^{\alpha\beta} = \begin{pmatrix} \alpha^i \\ -\beta^r \end{pmatrix} , \quad (\text{B.5})$$

where α^i and β^r are real antisymmetric 4×4 matrices that satisfy

$$\alpha_i \alpha_j = \epsilon_{ijk} \alpha_k - \delta_{ij} \mathbf{1} , \quad (\alpha^i)_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta} (\alpha^i)_{\gamma\delta} , \quad (\text{B.6})$$

$$\beta_r \beta_s = \epsilon_{rst} \beta_t - \delta_{rs} \mathbf{1} , \quad (\beta^r)_{\alpha\beta} = -\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta} (\beta^r)_{\gamma\delta} .$$

Further useful identities are

$$(\alpha^i)_{\alpha\beta} (\alpha^i)_{\gamma\delta} = \delta_{\alpha\gamma} \delta_{\beta\delta} - \delta_{\alpha\delta} \delta_{\beta\gamma} + \epsilon_{\alpha\beta\gamma\delta} , \quad (\text{B.7})$$

$$\epsilon^{ijk} (\alpha^j)_{\alpha\beta} (\alpha^k)_{\gamma\delta} = \delta_{\beta\gamma} (\alpha^i)_{\alpha\delta} + 3 \text{ more} , \quad (\text{B.8})$$

$$(\beta^r)_{\alpha\beta} (\beta^r)_{\gamma\delta} = \delta_{\alpha\gamma} \delta_{\beta\delta} - \delta_{\alpha\delta} \delta_{\beta\gamma} - \epsilon_{\alpha\beta\gamma\delta} , \quad (\text{B.9})$$

$$\epsilon^{trs} (\beta^r)_{\alpha\beta} (\beta^s)_{\gamma\delta} = \delta_{\beta\gamma} (\beta^t)_{\alpha\delta} + 3 \text{ more} . \quad (\text{B.10})$$

Using the above relations and recalling that \mathcal{V} is unimodular, it is simple to verify that

$$L_I^I L_J^B \eta_{AB} = \eta_{IJ} , \quad L_I^A L_J^B \eta^{IJ} = \eta^{AB} . \quad (\text{B.11})$$

As a further check, let us compare the potential

$$V = \frac{1}{4}e^{-\sigma} \left(C^{ir} C_{ir} - \frac{1}{9}C^2 \right) \quad (\text{B.12})$$

for the $SO(4)$ gauged theory with that of [10] where it is represented in terms of the $SL(4, R)$ coset representative. To begin with, the function C can be written as

$$\begin{aligned} C &= -\frac{1}{\sqrt{2}} f_{IJ}^K L_i^I L_j^J L_{Kk} \epsilon^{ijk} , \\ &= -\frac{1}{64\sqrt{2}} f_{IJK} \left(\mathcal{V} \Gamma^I \eta_i \mathcal{V} \right) \left(\mathcal{V} \Gamma^J \eta_j \mathcal{V} \right) \left(\mathcal{V} \Gamma^K \eta_k \mathcal{V} \right) \epsilon^{ijk} \\ &= \frac{1}{8\sqrt{2}} f_{IJK} \left[(\Gamma^{IJK})_{\alpha\beta} T^{\alpha\beta} + (\Gamma^{IJK})^{\alpha\beta} T_{\alpha\beta} \right] , \end{aligned} \quad (\text{B.13})$$

where

$$T_{\alpha\beta} = \mathcal{V}_\alpha^R \mathcal{V}_\beta^S \delta_{RS} , \quad T^{\alpha\beta} = \mathcal{V}_R^\alpha \mathcal{V}_S^\beta \delta_{RS} . \quad (\text{B.14})$$

In the last step we have used (B.8). In fact, the expression (3.20) is valid for any gauging, notwithstanding the fact that the $SO(4)$ invariant tensor δ_{RS} occurs in (B.14). However, only for $SO(4)$ gauging in which the f_{IJK} refers to the $SO(4)$ structure constants, (3.20) simplifies to give a direct relation between C and $T = T^{\alpha\beta} \delta_{\alpha\beta}$ that is manifestly $SO(4)$ invariant, as will be shown below. To obtain a similar relation for gaugings other than $SO(4)$, for example $SO(2, 2)$, we would need to construct the Γ and η matrices in a $SO(2, 1) \times SO(2, 1)$ basis with suitable changes in (B.6). In that case, the $SO(2, 2)$ invariant tensor η_{RS} would replace the $SO(4)$ invariant tensor δ_{RS} in (B.14) and we could get a manifestly $SO(2, 2)$ invariant direct relation between C and T .

In the case of $SO(4)$ gauging we have $f_{IJK} = (\epsilon_{ijk}, -\epsilon_{rst})$. Using this in (3.20) we find that the ϵ_{ijk} term gives a contribution of the form $(\delta_{\alpha\beta} T^{\alpha\beta} + \delta^{\alpha\beta} T_{\alpha\beta})$, while the ϵ_{rst} term gives a contribution of the form $(\delta_{\alpha\beta} T^{\alpha\beta} - \delta^{\alpha\beta} T_{\alpha\beta})$. The $\delta^{\alpha\beta} T_{\alpha\beta}$ contributions cancel and we are left with

$$C = -\frac{3}{2\sqrt{2}} T , \quad T \equiv T^{\alpha\beta} \delta_{\alpha\beta} . \quad (\text{B.15})$$

Similarly, it follows from the definition of C^{ir} and the orthogonality relations satisfied by L_I^A that

$$C^{ir} C_{ir} = f_{IJK} f_{MN}^K L_i^I L_j^J L_i^M L_j^N + \frac{1}{3} C^2 . \quad (\text{B.16})$$

Thus, it suffices to compute

$$\begin{aligned} f_{IJK} f_{MN}^K L_i^I L_j^J L_i^M L_j^N &= -\frac{1}{4} \left(\mathcal{V} \Gamma^i \eta^j \mathcal{V} \right) (\mathcal{V} \Gamma_i \eta_j \mathcal{V}) + 6 \\ &= \frac{1}{2} T_{RS} T^{RS} - \frac{1}{2} T^2 . \end{aligned} \quad (\text{B.17})$$

Using the results (B.15), (B.16) and (B.17) in (B.12), we find

$$V = \frac{1}{8} e^{-\sigma} \left(T_{RS} T^{RS} - \frac{1}{2} T^2 \right) , \quad (\text{B.18})$$

which agrees with the result of [10].

In the case of $Sp(1)_R$ gauged $6D$ supergravity obtained from the $SO(3, 1)$ gauged supergravity in $7D$, i.e. model II in section 3.2, we have $f_{IJK} = (-\epsilon_{rst}, -\epsilon_{ijr})$, where ϵ_{ijr} is totally antisymmetric and $\epsilon_{124} = \epsilon_{235} = \epsilon_{316} = 1$. For this case, the C -function has a more complicated form in terms of the $SL(4, R)$ coset representative \mathcal{V} . However, setting the scalar fields equal to zero, which is required for model II at hand, \mathcal{V} becomes a unit

matrix and the C -function vanishes. This is easily seen in the first line of (3.20), while it can be seen from the last line of (3.20) by noting that the ϵ_{rst} term gives the contribution $(\delta_{\alpha\beta}T^{\alpha\beta} - \delta^{\alpha\beta}T_{\alpha\beta})$, and the ϵ_{rsi} term give the structure $(\vec{\alpha} \cdot \vec{\beta})_{\alpha\beta}T^{\alpha\beta} + (\vec{\alpha} \cdot \vec{\beta})^{\alpha\beta}T_{\alpha\beta}$, where $\vec{\beta}$ refers to β_{r-3} , both of which vanish when \mathcal{V} is taken to be a unit matrix. In the second term this is due to the fact that $\vec{\alpha} \cdot \vec{\beta}$ is traceless.

C Dualization of the Axions in the Ungauged 6D Model

The $(p+3)$ axionic scalars occurring in the 6D Lagrangian $\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F$ with \mathcal{L}_B and \mathcal{L}_F given in (4.3) and (4.4) can be dualized to 4-form potentials with tensor gauge symmetry straightforwardly. Start by adding the suitable total derivative term to this Lagrangian to define

$$\mathcal{L}_4 = \mathcal{L}_B + \mathcal{L}_F + \frac{1}{5!\sqrt{6}} \epsilon^{\mu_1 \dots \mu_6} \left(-H_{\mu_1 \dots \mu_5}^i \mathcal{P}_{\mu_6}^i + H_{\mu_1 \dots \mu_5}^r \mathcal{P}_{\mu_6}^r \right) e^{-\varphi} , \quad (\text{C.1})$$

where the definitions (4.9) are to be used without the gauge coupling constants. Recalling that (2.4) holds, the Φ^I field equation implies $dH_5^I = 0$ with $H_5^i = H_5^I L_I^i$ and $H_5^r = H_5^I L_I^r$, which means that locally

$$H_{\mu_1 \dots \mu_5}^I = 5 \partial_{[\mu_1} C_{\mu_2 \dots \mu_5]}^I , \quad I = 1, \dots, p+3 . \quad (\text{C.2})$$

Solving for $(\mathcal{P}_\mu^i, \mathcal{P}_\mu^r)$ gives

$$\begin{aligned} \mathcal{P}_\mu^i &= \frac{\sqrt{2}}{5!} \epsilon^{\mu\nu_1 \dots \nu_5} H_{\nu_1 \dots \nu_5}^i - \bar{\psi} \gamma^\nu \gamma_\mu \sigma^i \psi_\nu - \frac{1}{2} X_\mu^i , \\ \mathcal{P}_\mu^r &= -\frac{\sqrt{2}}{5!} \epsilon^{\mu\nu_1 \dots \nu_5} H_{\nu_1 \dots \nu_5}^r + i \bar{\psi}^r \gamma^\nu \gamma_\mu \sigma^i \psi_\nu - 2i X_\mu^r . \end{aligned} \quad (\text{C.3})$$

Substituting these back into the Lagrangian (C.1), we get

$$\begin{aligned} \mathcal{L}_4 &= \mathcal{L}' - \frac{1}{2 \times 5!} e^{-2\varphi} H_{\mu_1 \dots \mu_5}^i H^{i\mu_1 \dots \mu_5} - \frac{1}{2 \times 5!} e^{-2\varphi} H_{\mu_1 \dots \mu_5}^r H^{r\mu_1 \dots \mu_5} \\ &\quad - \frac{1}{5! \sqrt{2}} e^{-\varphi} \epsilon^{\mu\nu_1 \dots \nu_5} H_{\nu_1 \dots \nu_5}^i \left(\bar{\psi} \gamma^\nu \gamma_\mu \sigma^i \psi_\nu + \frac{1}{2} X_\mu^i \right) \\ &\quad - \frac{1}{5! \sqrt{2}} e^{-\varphi} \epsilon^{\mu\nu_1 \dots \nu_5} H_{\nu_1 \dots \nu_5}^r \left(i \bar{\psi}^r \gamma^\nu \gamma_\mu \sigma^i \psi_\nu - 2i X_\mu^r \right) , \end{aligned} \quad (\text{C.4})$$

where \mathcal{L}' is the $(\mathcal{P}_\mu^i, \mathcal{P}_\mu^r)$ independent part of $\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F$ with \mathcal{L}_B and \mathcal{L}_F given in (4.3) and (4.4). Thus, we have $\mathcal{L}_4 = \mathcal{L}_{4B} + \mathcal{L}_{4F}$ with

$$\begin{aligned} e^{-1} \mathcal{L}_{4B} &= \frac{1}{4} R - \frac{1}{4} (\partial_\mu \sigma)^2 - \frac{1}{12} e^{2\sigma} G_{\mu\nu\rho} G^{\mu\nu\rho} - \frac{1}{8} e^\sigma F_{\mu\nu}^{r'} F^{\mu\nu r'} \\ &\quad - \frac{1}{4} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{8} P_\mu^{ir} P_{ir}^\mu - \frac{1}{2 \times 5!} e^{-2\varphi} a_{IJ} H_{\mu_1 \dots \mu_5}^I H^{J\mu_1 \dots \mu_5} , \end{aligned} \quad (\text{C.5})$$

$$\begin{aligned}
e^{-1}\mathcal{L}_F &= -\frac{i}{2}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{i}{2}\bar{\chi}\gamma^\mu D_\mu\chi - \frac{i}{2}\bar{\lambda}^{r'}\gamma^\mu D_\mu\lambda_{r'} \\
&\quad - \frac{i}{2}\bar{\psi}\gamma^\mu D_\mu\psi - \frac{i}{2}\bar{\psi}^r\gamma^\mu D_\mu\psi^r - \frac{i}{2}\bar{\chi}\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu\sigma \\
&\quad - \frac{1}{2}\bar{\psi}^r\gamma^\mu\gamma^\nu\sigma_i\psi_\mu P_\nu^{ir} + \frac{i}{2}\bar{\psi}\gamma^\mu\gamma^\nu\psi_\mu\partial_\nu\varphi \\
&\quad - \frac{1}{5!\sqrt{2}}e^{-\varphi}H_{\mu_1\cdots\mu_5}^I\left(\bar{\psi}\gamma^\nu\gamma_{\mu_1\cdots\mu_5}\sigma^i\psi_\nu L_I^i + \bar{\psi}^r\gamma^\nu\gamma_{\mu_1\cdots\mu_5}\sigma^i\psi_\nu L_I^r\right) \\
&\quad + \frac{i}{24}e^\sigma G_{\mu\nu\rho}X^{\mu\nu\rho} - \frac{i}{4}e^{\sigma/2}F_{\mu\nu}^{r'}X_{r'}^{\mu\nu} + \frac{1}{2\sqrt{2}\times 5!}e^{-\varphi}H_{\mu_1\cdots\mu_5}^IX_I^{\mu_1\cdots\mu_5}, \quad (C.6)
\end{aligned}$$

where the structure constants (hence the C -functions as well) are to be set to zero in the definitions (4.7) and (4.8), and

$$\begin{aligned}
X^{\mu\nu\rho} &= \bar{\psi}^\lambda\gamma_{[\lambda}\gamma^{\mu\nu\rho}\gamma_{\tau]}\psi^\tau + 2\bar{\psi}_\lambda\gamma^{\mu\nu\rho}\gamma^\lambda\chi - 2\bar{\chi}\gamma^{\mu\nu\rho}\chi + \bar{\lambda}^{r'}\gamma^{\mu\nu\rho}\lambda_{r'} + \bar{\psi}^r\gamma^{\mu\nu\rho}\psi_r + \bar{\psi}\gamma^{\mu\nu\rho}\psi, \\
X_{r'}^{\mu\nu} &= \bar{\psi}_\rho\gamma^{\mu\nu}\gamma^\rho\lambda_{r'} + \bar{\chi}\gamma^{\mu\nu}\lambda_{r'}, \\
X_I^{\mu_1\cdots\mu_5} &= L_I^i(\bar{\psi}^\rho\gamma_{[\rho}\gamma^{\mu_1\cdots\mu_5}\gamma_{\tau]}\sigma_i\psi^\tau + 2\bar{\chi}\gamma^{\mu_1\cdots\mu_5}\sigma_i\chi + \bar{\lambda}^{r'}\gamma^{\mu_1\cdots\mu_5}\sigma_i\lambda_{r'} \\
&\quad - \bar{\psi}^r\gamma^{\mu_1\cdots\mu_5}\sigma_i\psi_r - \bar{\psi}\gamma^{\mu_1\cdots\mu_5}\sigma_i\psi) - 4iL_I^r\bar{\psi}\gamma^{\mu_1\cdots\mu_5}\psi_r. \quad (C.7)
\end{aligned}$$

The action is invariant under the following supersymmetry transformations:

$$\begin{aligned}
\delta e_\mu^m &= i\bar{\epsilon}\gamma^m\psi_\mu, \\
\delta\psi_\mu &= D_\mu\epsilon - \frac{1}{24}e^\sigma\gamma^{\rho\sigma\tau}\gamma_\mu G_{\rho\sigma\tau}\epsilon - \frac{i}{5!\sqrt{6}}e^{-\varphi}\gamma^{\mu\nu_1\cdots\nu_5}H_{\nu_1\cdots\nu_5}^i\epsilon, \\
\delta\chi &= -\frac{1}{2}\gamma^\mu\partial_\mu\sigma\epsilon - \frac{1}{12}e^\sigma\gamma^{\rho\sigma\tau}G_{\rho\sigma\tau}\epsilon, \\
\delta B_{\mu\nu} &= ie^{-\sigma}\left(\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]} + \frac{1}{2}\bar{\epsilon}\gamma_{\mu\nu}\chi\right) - A_{[\mu}^{r'}\delta A_{\nu]}^{r'}, \\
\delta\sigma &= -i\bar{\epsilon}\chi, \\
\delta A_\mu^{r'} &= ie^{-\sigma/2}\bar{\epsilon}\gamma_\mu\lambda^{r'}, \\
\delta\lambda^{r'} &= -\frac{1}{4}e^{\sigma/2}\gamma^{\mu\nu}F_{\mu\nu}^{r'}\epsilon, \\
\delta C_{\mu_1\cdots\mu_4}^I &= -\frac{1}{\sqrt{2}}\left(\bar{\epsilon}\gamma_{\mu_1\cdots\mu_4}\sigma^i\psi + 4\bar{\epsilon}\gamma_{[\mu_1\cdots\mu_3}\sigma^i\psi_{\mu_4]}\right)L_I^i - \frac{i}{\sqrt{2}}\bar{\epsilon}\gamma_{\mu_1\cdots\mu_4}\psi^r L_r^I, \\
L_i^I\delta L_I^r &= -\bar{\epsilon}\sigma_i\psi^r, \\
\delta\varphi &= i\bar{\epsilon}\psi, \\
\delta\psi &= \frac{1}{2}\gamma^\mu\partial_\mu\varphi\epsilon + \frac{i}{5!\sqrt{2}}e^{-\varphi}\gamma^{\mu_1\cdots\mu_5}H_{\mu_1\cdots\mu_5}^i\epsilon,
\end{aligned}$$

$$\delta\psi^r = \frac{i}{2}\gamma^\mu P_\mu^{ir}\sigma_i - \frac{i}{5!\sqrt{2}}e^{-\varphi}\gamma^{\mu_1\cdots\mu_5}H_{\mu_1\cdots\mu_5}^r\epsilon. \quad (\text{C.8})$$

The supersymmetry transformation rule for $C_{\mu_1\cdots\mu_4}^I$ is derived from the requirement of supercovariance of (C.3), which requires the cancellation of the $\partial_\mu\epsilon$ terms.

D The Iwasawa Decomposition of $SO(p, q)$

We begin with the Iwasawa decomposition of the $SO(n+3, 3)$ algebra as $g = h \oplus a \oplus n$ where

$$\begin{aligned} h : & \quad X_{ij}, Y_{ij}, Z_{ir}, T_{rs}, \\ a : & \quad H_i, \\ n : & \quad E_i^j, V^{ij}, U_{ir}, \quad i > j. \end{aligned} \quad (\text{D.1})$$

Here $X = E - E^T$, $Y = V - V^T$, $Z = U - U^T$, together with the $SO(n)$ generators $T_{rs} = -T_{sr}$ form the maximal compact subalgebra $\{h\}$ of $SO(n+3) \times SO(3)$. Furthermore, $\{a\}$ are the noncompact Cartan generators and $\{n\}$ are the remaining noncompact generators of $SO(n+3, 3)$. The generators $a \oplus n$ form the solvable subalgebra of $SO(n+3, 3)$, and can be represented as (see, for example, [20])

$$\begin{aligned} \vec{H} &= \left(\begin{array}{c|c|c} \sum_i \vec{c}_i e_{ii} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & -\sum_i \vec{c}_i e_{ii} \end{array} \right), & E_i^j &= \left(\begin{array}{c|c|c} -e_{ji} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & e_{ij} \end{array} \right), \\ V^{ij} &= \left(\begin{array}{c|c|c} 0 & 0 & e_{ij} - e_{ji} \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \right), & U_r^i &= \left(\begin{array}{c|c|c} 0 & e_{ir} & 0 \\ \hline 0 & 0 & e_{ri} \\ \hline 0 & 0 & 0 \end{array} \right). \end{aligned} \quad (\text{D.2})$$

The maximal compact subalgebra generators are then represented as

$$\begin{aligned} X_{ij} &= \left(\begin{array}{c|c|c} e_{ij} - e_{ji} & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & e_{ij} - e_{ji} \end{array} \right), & Y_{ij} &= \left(\begin{array}{c|c|c} 0 & 0 & e_{ij} - e_{ji} \\ \hline 0 & 0 & 0 \\ \hline e_{ij} - e_{ji} & 0 & 0 \end{array} \right), \\ Z_{ir} &= \left(\begin{array}{c|c|c} 0 & e_{ir} & 0 \\ \hline -e_{ri} & 0 & e_{ri} \\ \hline 0 & -e_{ir} & 0 \end{array} \right), & T_{rs} &= \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & e_{rs} - e_{sr} & 0 \\ \hline 0 & 0 & 0 \end{array} \right). \end{aligned} \quad (\text{D.3})$$

Each e_{ab} is defined to be a matrix of the appropriate dimensions that has zeros in all its entries except for a 1 in the entry at row a and column b . These satisfy the matrix product rule $e_{ab} e_{cd} = \delta_{bc} e_{ad}$.

The solvable subalgebra of $SO(n+3, 3)$ has the nonvanishing commutators

$$\begin{aligned}
[\vec{H}, E_i^j] &= \vec{b}_{ij} E_i^j, & [\vec{H}, V^{ij}] &= \vec{a}_{ij} V^{ij}, & [\vec{H}, U_r^j] &= \vec{c}_i U_r^i, \\
[E_i^j, E_k^\ell] &= \delta_k^j E_i^\ell - \delta_i^\ell E_k^j, \\
[E_i^j, V^{k\ell}] &= -\delta_i^k V^{j\ell} - \delta_i^\ell V^{kj}, & [E_i^j, U_r^k] &= -\delta_i^k U_r^j, \\
[U_r^i, U_s^j] &= \delta_{rs} V^{ij},
\end{aligned} \tag{D.4}$$

where the structure constants are given by

$$\vec{b}_{ij} = \sqrt{2} (-\vec{e}_i + \vec{e}_j), \quad \vec{a}_{ij} = \sqrt{2} (\vec{e}_i + \vec{e}_j), \quad \vec{c}_i = \sqrt{2} \vec{e}_i. \tag{D.5}$$

The nonvanishing commutation rules of the maximal compact subalgebra $SO(n+3) \oplus SO(3)$ are

$$\begin{aligned}
[X_{ij}, X_{k\ell}] &= \delta_{jk} X_{i\ell} + 3 \text{ perms}, & [T_{pq}, T_{rs}] &= \delta_{qr} T_{ps} + 3 \text{ perms}, \\
[X_{ij}, Y_{k\ell}] &= \delta_{jk} Y_{i\ell} + 3 \text{ perms}, & [X_{ij}, Z_{kr}] &= \delta_{jk} Z_{ir} - \delta_{ik} Z_{jr}, \\
[Y_{ij}, Y_{k\ell}] &= \delta_{jk} X_{i\ell} + 3 \text{ perms}, & [T_{pq}, Z_{ir}] &= \delta_{qr} Z_{ip} - \delta_{pr} Z_{iq}, \\
[Z_{ir}, Z_{js}] &= -\delta_{rs} X_{ij} + \delta_{rs} Y_{ij} - 2\delta_{ij} T_{rs}, & [Y_{ij}, Z_{kr}] &= -\delta_{jk} Z_{ir} + \delta_{ik} Z_{jr}.
\end{aligned} \tag{D.6}$$

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